

2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

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2.1 COHOMOLOGY RINGS OF 6-MANIFOLDS

Let (r, H, w, τ, F, p) be a system of invariants as in section 1; recall that it is admissible iff for every $W \in H$, $T \in H^\vee$ with $\bar{W} = w(\text{mod } 2)$, $\bar{T} \equiv \tau(\text{mod } 2)$ the following congruence holds:

$$(*) \quad W^3 \equiv (p + 24T)(W) \pmod{48}.$$

LEMMA 1. (r, H, w, τ, F, p) is admissible if and only if there exist $W_o \in H$, $T_o \in H^\vee$ with $\bar{W}_o \equiv w(\text{mod } 2)$, $\bar{T}_o \equiv \tau(\text{mod } 2)$, such that

- i) $W_o^3 \equiv (p + 24T_o)(W_o) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2 W_o + 3x W_o^2 \pmod{24} \quad \forall x \in H$.

Proof. Obvious since the set of integral lifts of w is a coset $W_o + 2H$.

DEFINITION 3. Let $F \in S^3 H^\vee$ be a symmetric trilinear form on a finitely generated free abelian group H . An element $W \in H$ is characteristic for F iff

$$(**) \quad x \cdot y \cdot (x + y + W) \equiv 0 \pmod{2} \quad \forall x, y \in H.$$

LEMMA 2. $W \in H$ is a characteristic element for $F \in S^3 H^\vee$ if and only if the function $l_W: H \rightarrow \mathbf{Z}$, $l_W(x) := 4x^3 + 6x^2 W + 3x W^2$ is linear in x modulo 24.

Proof. $l_W(x + y) = l_W(x) + l_W(y) + 12(x^2 y + x y^2 + x y W)$, whence the assertion.

The existence of characteristic elements is a necessary and sufficient condition for a cubic form $F \in S^3 H^\vee$ to be realizable by a manifold. In fact, we have:

PROPOSITION 2. A given cubic form $F \in S^3 H^\vee$ on a finitely generated free abelian group H is realizable as cup-form of a 1-connected, closed, oriented, 6-dimensional manifold with torsion-free homology if and only if it possesses a characteristic element.

Proof. If (r, H, w, τ, F, p) is an admissible system of invariants, and $W_o \in H$ any integral lift of w , then we have $p(x) \equiv 4x^3 + 6x^2 W_o + 3x W_o^2 \pmod{24} \quad \forall x \in H$, i.e. the function $l_{W_o}: H \rightarrow \mathbf{Z}$ is linear modulo 24, and W_o is therefore characteristic for F . Conversely, suppose $W_o \in H$ is a characteristic element for a cubic form $F \in S^3 H^\vee$; let $w := \bar{W}_o(\text{mod } 2)$, $r := 0$.

By the main lemma we have to construct linear forms $p, T \in H^\vee$, such that

- i) $W_\circ^3 \equiv (p + 24T)(W_\circ) \pmod{48}$
- ii) $p(x) \equiv 4x^3 + 6x^2 W_\circ + 3x W_\circ^2 \pmod{24} \quad \forall x \in H$.

The function $l_{W_\circ}: H \rightarrow \mathbf{Z}, l_{W_\circ}(x) = 4x^3 + 6x^2 W_\circ + 3x W_\circ^2$ is linear modulo 24 since $W_\circ$ is a characteristic element for F : we therefore choose a linear form $p_\circ \in H^\vee$ with $p_\circ(x) \equiv l_{W_\circ}(x) \pmod{24} \quad \forall x \in H$. Substituting $x = W_\circ$ we find $p_\circ(W_\circ) \equiv 13 W_\circ^3 \pmod{24}$; but since $W_\circ$ is characteristic we have $W_\circ^3 \equiv 0 \pmod{2}$, thus $p_\circ(W_\circ) \equiv W_\circ^3 \pmod{24}$. Write $p_\circ(W_\circ) = W_\circ^3 + 24k$ for some $k \in \mathbf{Z}$.

case 1) $k \equiv 0 \pmod{2}$: define $p := p_\circ, T := 0$.

case 2) $k \equiv 1 \pmod{2}$: we must find a linear form $T_\circ \in H^\vee$ with $T_\circ(W_\circ) \equiv 1 \pmod{2}$; clearly this can be done if and only if $W_\circ$ is not divisible by 2. If $W_\circ$ were divisible by 2, $W_\circ = 2V_\circ$ for some $V_\circ \in H$, then $2p_\circ(V_\circ) = p_\circ(W_\circ) = W_\circ^3 + 24k = 8V_\circ^3 + 24k$ would give $p_\circ(V_\circ) = 4V_\circ^3 + 12k$; then, using $p_\circ(V_\circ) \equiv 4V_\circ^3 + 6V_\circ^2 W_\circ + 3V_\circ W_\circ^2 \pmod{24}$ we would find $k \equiv 0 \pmod{2}$, which is not the case by assumption.

This shows that $F \in S^3 H^\vee$ is realizable by a topological manifold with Pontrjagin class $p_\circ$ and non-vanishing triangulation obstruction $\tau_\circ := \bar{T}_\circ \pmod{2}$. In order to realize F by a smooth manifold, one can take $p := p_\circ + 24T_\circ$, and $\tau := 0$.

REMARK 3. The topological counterpart of the existence of a characteristic element for a given cubic form $F \in S^3 H^\vee$ is the existence of a mod-2 Steenrod-algebra structure, which is a necessary condition for a ring to be a cohomology ring.

The existence and the classification of characteristic elements for a given cubic form is essentially a linear algebra problem over $\mathbf{Z}_{/2}$. To see this, let $F \in S^3 H^\vee$ be a fixed cubic form on a finitely generated free abelian group H . Associated with F we have a linear map $F^t: H \rightarrow S^2 H^\vee$ sending an element $h \in H$ to the bilinear form $F^t(h): H \otimes H \rightarrow \mathbf{Z}, (x, y) \rightarrow x \cdot y \cdot h$. Let $\bar{H} := H/_{2H}$. $\bar{F} \in S^3 \bar{H}^\vee$ be the reductions of H and F modulo 2, and let $-: H \rightarrow \bar{H}$ be the natural epimorphism. The symmetric trilinear form $\bar{F}$ on the $\mathbf{Z}_{/2}$ -module $\bar{H}$ defines a natural symmetric bilinear form $q_{\bar{F}} \in S^2 \bar{H}^\vee$ given by $q_{\bar{F}}(\bar{x}, \bar{y}) := \bar{x} \cdot \bar{y} \cdot (\bar{x} + \bar{y})$.

LEMMA 3. $F \in S^3 H^\vee$ admits characteristic elements if and only if $q_{\bar{F}}$ lies in the image of $\bar{F}^t \in \text{Hom}_{\mathbf{Z}}(H, S^2 \bar{H}^\vee)$. The set of all characteristic elements for F is a coset of the form $W_\circ + \text{Ker}(\bar{F}^t)$.

Proof. W_o is characteristic for F if and only if $q_{\bar{F}} = \bar{F}^t(W_o)$.

In terms of a $\mathbf{Z}$ -basis $\{e_1, \dots, e_b\}$ for H the condition $q_{\bar{F}} \in \text{Im}(\bar{F}^t)$ translates into a simple rank condition over $\mathbf{Z}_{/2}$: the $\mathbf{Z}_{/2}$ -rank of the $b \times \binom{b+1}{2}$ -matrix A representing $\bar{F}^t$ must be equal to the $\mathbf{Z}_{/2}$ -rank of the matrix A extended by the column $(\bar{e}_i \cdot \bar{e}_j \cdot (\bar{e}_i + \bar{e}_j))_{1 \leq i \leq j \leq b}$

EXAMPLE 3. Let $H = \mathbf{Z}e_1 \oplus \mathbf{Z}e_2$ be free of rank 2, $F \in S^3 H^\vee$ given by $e_1^3 = a$, $e_1^2 e_2 = b$, $e_1 e_2^2 = c$, $e_2^3 = d$ with $a, b, c, d \in \mathbf{Z}$. The rank condition becomes

$$rk_2 \begin{bmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \\ \bar{b} & \bar{c} \end{bmatrix} = rk_2 \begin{bmatrix} \bar{a} & \bar{b} & \bar{0} \\ \bar{c} & \bar{d} & \bar{0} \\ \bar{b} & \bar{c} & \overline{b+c} \end{bmatrix}$$

2.2 HOMOTOPY TYPES WITH A GIVEN COHOMOLOGY RING

Our next task is to describe the set of oriented homotopy types of 1-connected, closed, oriented, 6-dimensional manifolds with a fixed torsion-free cohomology ring.

From Žubr's classification theorem we know that in algebraic terms this means the following: fix a non-negative integer r_o , a finitely generated free abelian group H_o , and a symmetric trilinear form $F_o \in S^3 H_o^\vee$ which admits characteristic elements.

Let $\mathcal{M}(r_o, H_o, F_o)$ be the set of 1-connected, closed, oriented, 6-dimensional manifolds X with $b_3(X) = 2r_o$, such that there exists an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$. Denote by $\text{Aut}(F_o)$ the subgroup of $\mathbf{Z}$ -automorphisms of H_o which leave $F_o \in S^3 H_o^\vee$ invariant; $\text{Aut}(F_o)$ acts on pairs $(w, [l]) \in \bar{H}_o \times H_o^\vee / {}_{48}H_o^\vee / {}_{U_{F_o}}$ in a natural way:

$$\gamma \cdot (w, [l]) := (\gamma(w), (\gamma^{-1})^*[l]) .$$

Let ${}_{\text{Aut}(F_o)} \backslash \bar{H}_o \times H_o^\vee / {}_{48}H_o^\vee / {}_{U_{F_o}}$ be the set of $\text{Aut}(F_o)$ -orbits.

A manifold X in $\mathcal{M}(r_o, H_o, F_o)$ and an isomorphism $\alpha: H_o \rightarrow H^2(X, \mathbf{Z})$ with $\alpha^* F_X = F_o$ yields a well-defined $\text{Aut}(F_o)$ -orbit:

$$(\alpha^{-1}(w_2(X)), \alpha^*[p_1(X) + 24T]) \text{ (modulo } \text{Aut}(F_o) \text{)},$$

where $T \in H^4(X, \mathbf{Z})$ is an arbitrary integral lifting of $\tau(X) \in H^4(X, \mathbf{Z}_{/2})$.

The set of oriented homotopy types $\mathcal{M}(r_o, H_o, F_o) / \simeq$ of manifolds in $\mathcal{M}(r_o, H_o, F_o)$ can now be described in the following way: