1. Hyperbolic 3-manifolds and Bloch groups

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1. Hyperbolic 3-manifolds and Bloch groups

We will use the upper half space model for the hyperbolic 3-space $\mathscr{H}^3$. Note that $\mathscr{H}^3$ has a standard compactification by adding a boundary $\partial\mathscr{H}^3 = \mathbf{C} \cup \{\infty\} = \mathbf{P}^1(\mathbf{C})$. The group of orientation preserving isometries of $\mathscr{H}^3$ is $PSL(2, \mathbf{C})$, which acts on the boundary $\partial\mathscr{H}^3$ via linear fractional transformations. Given any 4 ordered distinct points (a_0, a_1, a_2, a_3) on $\partial\mathscr{H}^3$, there exists a unique element in $PSL(2, \mathbf{C})$ that maps (a_0, a_1, a_2, a_3) to $(0, \infty, 1, z)$, where

$$z = \frac{(a_2 - a_1)(a_3 - a_0)}{(a_2 - a_0)(a_3 - a_1)} \in \mathbf{C} - \{0, 1\} ,$$

is the cross ratio of a_0, a_1, a_2, a_3. Given any *ideal* tetrahedron (vertices at the boundary) in $\mathscr{H}^3$ and an ordering of its vertices, we can associate to it the cross ratio of its vertices which is a complex number in $\mathbf{C} - \{0, 1\}$. If we change the ordering of the vertices by an even permutation (so the orientation of the ideal tetrahedron is not changed) we obtain 3 possible answers for the cross ratio: z, $1 - 1/z$, and $1/(1 - z)$.

There are several definitions in the literature of the Bloch group $\mathscr{B}(F)$ of an infinite field F. They are isomorphic with each other modulo torsion of order dividing 6, and they are strictly isomorphic for F algebraically closed (see [6] and [14] for a more detailed discussion). The most widely used are those of Dupont and Sah [6] and of Suslin [19]. We shall use a definition equivalent to Dupont and Sah's, since it is the most appropriate for questions involving $PGL_2(F)$. Suslin's definition is appropriate for questions involving $GL_2(F)$ and gives a group which naturally embeds in ours with quotient of exponent at most 2. However, since torsion is irrelevant to our current considerations, the precise definition we choose is, in fact, unimportant.

Our $\mathscr{B}(F)$ is defined as follows. Let $\mathbf{Z}(\mathbf{C} - \{0, 1\})$ denote the free group generated by points in $\mathbf{C} - \{0, 1\}$. Define the map

$$\mu : \mathbf{Z}(\mathbf{C} - \{0, 1\}) \to \wedge^2_{\mathbf{Z}}(\mathbf{C}^*) ,$$

$$\sum_i n_i [z_i] \mapsto \sum_i n_i z_i \wedge (1 - z_i) ,$$

and let $\mathscr{A}(F) = \operatorname{Ker}(2\mu)$.

The Bloch group $\mathscr{B}(F)$ is the quotient of $\mathscr{A}(F)$ by all instances of the following "5-term relation"

$$[x] - [y] + [y/x] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - y}{1 - x}\right] = 0 ,$$

where $x, y \in F - \{0, 1\}$. One checks easily that the map μ vanishes identically on this 5-term relation.

REMARK. The above 5-term relation is that of Suslin [19], which differs slightly from the following more widely used 5-term equation (first written down by Bloch-Wigner)

$$[x] - [y] + [y/x] - \left[\frac{1-y}{1-x}\right] + \left[\frac{1-y^{-1}}{1-x^{-1}}\right] = 0 .$$

These two different relations lead to mutually isomorphic $\mathscr{B}(\mathbf{C})$ via the map $[z] \mapsto [1/z]$. However, Suslin's 5-term relation seems more suitable for general F since it vanishes exactly under μ for an arbitrary F while the 5-term equation of Bloch-Wigner only vanishes up to 2-torsion [1]).

The geometric meaning of the 5-term equation is the following:

Let $a_0, \ldots, a_4 \in \mathbf{P}_1 = \mathscr{P}(\partial \mathscr{H}^3)$. Then the convex hull of these points in $\mathscr{H}^3$ can be decomposed into ideal tetrahedra in two ways, either as the union of the three ideal tetrahedra $\{a_1, a_2, a_3, a_4\}$, $\{a_0, a_1, a_3, a_4\}$ and $\{a_0, a_1, a_2, a_3\}$ or as the union of two ideal tetrahedra $\{a_0, a_2, a_3, a_4\}$ and $\{a_0, a_1, a_2, a_4\}$. Taking $(a_0, \ldots, a_4) = (0, \infty, 1, x, y)$, then the five terms in the above relation occur exactly as the five cross ratios of the corresponding ideal tetrahedra.

We discuss briefly how a hyperbolic 3-manifold determines an element in $\mathscr{B}(\mathbf{C})$. For more details, see [14].

It is straightforward for a closed hyperbolic 3-manifold M. Let $M = \mathscr{H}^3/\Gamma$, where Γ is a discrete subgroup of $PSL(2, \mathbf{C})$. The fundamental class of M in $H_3(M, \mathbf{Z})$ then gives rise to a class in $H_3(PSL_2(\mathbf{C}); \mathbf{Z})$ via the map

$$H_3(M, \mathbf{Z}) \cong H_3(\Gamma, \mathbf{Z}) \to H_3(PSL(2, \mathbf{C}), \mathbf{Z}) .$$

Fix a base point in $\mathbf{C} - \{0, 1\}$. Then via the natural action of $PSL(2, \mathbf{C})$ on $\mathbf{P}^1(\mathbf{C})$, any 4-tuple of elements in $PSL(2, \mathbf{C})$ gives a 4-tuple of elements in $\mathbf{C}$ which then gives rise to an element in $\mathbf{C}$ via the cross ratio. This induces a well defined homomorphism

$$H_3(PSL(2, \mathbf{C}), \mathbf{Z}) \to \mathscr{B}(\mathbf{C}) .$$

[1]) One could therefore give an alternate definition of the Bloch group by replacing 2μ by μ in the definition of $\mathscr{A}(F)$. This gives a group that lies between Suslin's Bloch group and the one defined here. Although this may seem more natural, the current definition is suggested by homological considerations. If F is algebraically closed the factor 2 makes no difference.

Although the cross ratio depends on the particular choice of the base point, the above homomorphism is independent of this choice (see [6, (4.10)]).

In the cusped case, we note that by [7], each cusped hyperbolic 3-manifold M has an ideal triangulation

$$M = \Delta_1 \cup \cdots \cup \Delta_n .$$

Let z_i be the cross ratio of any ordering of the vertices of Δ_i consistent with its orientation. The following relation was first discovered by W. Thurston (unpublished — see the remark in Zagier [21]; for a proof, see [14]):

$$\sum_i z_i \wedge (1 - z_i) = 0, \quad \text{in } \wedge^2_{\mathbf{Z}}(\mathbf{C}^*) .$$

This relation is independent of the particular choice of the ordering of the vertices of the Δ_i's.

From the definition of Bloch groups, it is thus clear that an ideal decomposition of a cusped hyperbolic 3-manifold determines an element $\beta(M) = \sum_i [z_i]$ in the Bloch group $\mathcal{B}(\mathbf{C})$. That such an element is independent of the choice of the ideal triangulation is proved in [14].

In fact, what we proved in [14] is stronger. It includes the following, which is what is needed for the purpose of this paper.

THEOREM 1.1. *Every hyperbolic 3-manifold* M *gives a well defined element* $\beta(M)$ *in the Bloch group* $\mathcal{B}(\mathbf{C}) \otimes \mathbf{Q}$ *which is the image of a well defined element* $\beta_k(M)$ *in* $\mathcal{B}(k) \otimes \mathbf{Q}$ *where* k *is the invariant trace field of* M. □

2. Borel's Theorem

The Bloch group of an infinite field F is closely related to the third algebraic K-group of F (denoted by $K_3(F)$). There exists a natural map

$$\mathcal{B}(F) \to K_3(F)$$

due to Bloch [2]. In particular, if F is a number field, then there is the isomorphism

$$\mathcal{B}(F) \otimes \mathbf{Q} \cong K_3(F) \otimes \mathbf{Q} .$$

For the precise relationship between the Bloch group and K_3 of an infinite field, see the work of Suslin [19].

In [3], Borel generalized the classical Dirichlet Unit Theorem in classical number theory. The generalization started with the observation that for a number field F, the unit group of the ring of integers $\mathcal{O}_F$ is non-other than