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RATIONALITY PROBLEMS FOR K-THEORY AND CHERN-SIMONS INVARIANTS OF HYPERBOLIC 3-MANIFOLDS

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This paper makes certain observations regarding some conjectures of Milnor and Ramakrishnan in hyperbolic geometry and algebraic K-theory. As a consequence of our observations, we obtain new results and conjectures regarding the rationality and irrationality of Chern-Simons invariants of hyperbolic 3-manifolds.

In this paper, by a *hyperbolic 3-manifold*, we shall mean a complete, oriented hyperbolic 3-manifold with finite volume. So a hyperbolic 3-manifold is compact or has finitely many cusps as its ends (see e.g. [20]). By a *cusped manifold* we shall mean a non-compact hyperbolic 3-manifold.

A hyperbolic 3-manifold M is a quotient of the hyperbolic 3-space $\mathcal{H}^3$ by a discrete subgroup Γ of $PSL_2(\mathbf{C})$ with finite covolume. The isometry class of M determines the discrete subgroup up to conjugation. To each subgroup Γ of $PSL_2(\mathbf{C})$, we can associate the *trace field* of Γ, that is, the subfield of $\mathbf{C}$ generated by traces of all elements in Γ. The trace field clearly depends on the conjugacy class of Γ, so one can define it to be the trace field of the hyperbolic 3-manifold. However, the trace field is not an invariant of commensurability class of Γ, although it is not far removed. There is a notion of *invariant trace field* due to Alan Reid (see [17]) which is a subfield of trace field and does give an invariant of commensurability classes. Let $\Gamma^{(2)}$ be the subgroup of Γ generated by squares of elements of Γ. The invariant trace field $k(M)$ of M is defined to be the trace field of $\Gamma^{(2)}$. We will use invariant trace fields throughout this paper. We want to emphasize here that each invariant trace field is a number field together with a *specific embedding in* $\mathbf{C}$.

Recall that a number field is totally real if all its embeddings into $\mathbf{C}$ have image in $\mathbf{R}$ and totally imaginary if none of its embeddings has image

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in $\mathbf{R}$. A *CM-field* is a number field with complex multiplication, i.e., it is a totally imaginary quadratic extension of a totally real number field. A familiar class of CM-fields is the class of cyclotomic fields. We will also use a slightly more general notion. We say that an embedding $\sigma: F \hookrightarrow \mathbf{C}$ of a number field F is a *CM-embedding* if $\sigma(F)$, as a subfield of $\mathbf{C}$, is an imaginary quadratic extension of a totally real field. So F is a CM-field if and only if all its embeddings are CM-embeddings.

The Chern-Simons invariant of a compact $(4n-1)$-dimensional Riemannian manifold is an obstruction to conformal immersion of the Riemannian manifold in Euclidean space [4]. For hyperbolic 3-manifolds Meyerhoff [9] extended the definition to allow manifolds with cusps. The Chern-Simons invariant $CS(M)$ of a hyperbolic 3-manifold M is an element in $\mathbf{R}/\pi^2\mathbf{Z}$. It is *rational* (also called *torsion*) if it lies in $\pi^2\mathbf{Q}/\pi^2\mathbf{Z}$. The main application of our paper to Chern Simons invariants is the following theorem.

THEOREM A. *The Chern-Simons invariant $CS(M)$ of a hyperbolic 3-manifold is rational if the associated embedding of the invariant trace field $k(M)$ of M is a CM-embedding.*

Numerical evidence suggests that the Chern-Simons invariant of a hyperbolic 3-manifold is usually irrational when the Theorem A does not apply (though irrationality of Chern-Simons invariant — and of volume — has not been proved for any example). In particular, our results lead to the following explicit irrationality conjecture.

CONJECTURE. *If the invariant trace field $k = k(M)$ satisfies $k \cap \bar{k} \subset \mathbf{R}$ then $CS(M)$ is irrational. In particular, $CS(M)$ is irrational if $k(M)$ has odd degree over $\mathbf{Q}$.*

The invariant trace field k of an *arithmetic* hyperbolic 3-manifold has just one complex place (cf. e.g., [13]), so it either satisfies $k = \bar{k}$ and is CM-embedded or it satisfies $k \cap \bar{k} \subset \mathbf{R}$. Thus, in this case the conjecture would say that rationality or irrationality of the Chern-Simons invariant is completely determined by whether or not $k = \bar{k}$. (Recently Reznikov [18] claimed rationality of the Chern-Simons invariant for *any* compact arithmetic hyperbolic 3-manifold. However, his concept of "arithmetic" is what is usually called "real-arithmetic" and implies that k is CM-embedded, so his result follows from Theorem A. But it also has an easier proof: a real-arithmetic manifold has a cover with an orientation reversing isometry.)

Theorem A will follow from a more general result about Bloch groups. In [14], we showed that the Chern-Simons invariant $CS(M)$ is determined modulo rationals by an element $\beta(M)$ in the Bloch group $\mathscr{B}(k(M))$ (with $k(M)$ replaced by $\mathbf{C}$ this was known in the compact case by Dupont [5]).

We will prove a general result on the Bloch group of a number field F which is embedding into $\mathbf{C}$ as a non-real subfield that is stable under the complex conjugation. If F is a non-real Galois field, then any embedding has this property.

Complex conjugation then induces an involution on F and hence on $\mathscr{B}(F)$. Let $\mathscr{B}(F) \otimes \mathbf{Q} = \mathscr{B}_+(F) \oplus \mathscr{B}_-(F)$ be the splitting into ± 1 eigenspaces for such an involution. We emphasize that even in the Galois case, the involution, and therefore also this splitting, depends very much on the specific embedding of F into $\mathbf{C}$ that one uses (though if F happens to be a CM-field, then complex conjugation commutes with all the embeddings of F into $\mathbf{C}$ so the decomposition is independent of the choice of embedding). The following theorem gives formulas for the ranks of $\mathscr{B}_+(F)$ and $\mathscr{B}_-(F)$.

THEOREM B. *Assume F is a fixed non-real subfield of $\mathbf{C}$ which is finite over $\mathbf{Q}$ and stable under complex conjugation in $\mathbf{C}$. Let r_2 be the number of pairs of conjugate complex embeddings of F, and let r_2' be the number of pairs $\tau, \bar{\tau}$ of complex embeddings of F such that*

$$\tau(\bar{x}) = \bar{\tau}(x) \quad \forall x \in F \subset \mathbf{C}\,.$$

Then the following equations hold:

$$\operatorname{rank}(\mathscr{B}_-(F)) = \frac{1}{2}(r_2 + r_2')\,,$$

$$\operatorname{rank}(\mathscr{B}_+(F)) = \frac{1}{2}(r_2 - r_2')\,.$$

In Theorem 3.1 we describe the situation when F is not stable under conjugation.

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1. Hyperbolic 3-manifolds and Bloch groups

We will use the upper half space model for the hyperbolic 3-space $\mathscr{H}^3$. Note that $\mathscr{H}^3$ has a standard compactification by adding a boundary $\partial \mathscr{H}^3 = \mathbf{C} \cup \{\infty\} = \mathbf{P}^1(\mathbf{C})$. The group of orientation preserving isometries of $\mathscr{H}^3$ is $PSL(2, \mathbf{C})$, which acts on the boundary $\partial \mathscr{H}^3$ via linear fractional transformations. Given any 4 ordered distinct points (a_0, a_1, a_2, a_3) on $\partial \mathscr{H}^3$, there exists a unique element in $PSL(2, \mathbf{C})$ that maps (a_0, a_1, a_2, a_3) to $(0, \infty, 1, z)$, where

$$z = \frac{(a_2 - a_1)(a_3 - a_0)}{(a_2 - a_0)(a_3 - a_1)} \in \mathbf{C} - \{0, 1\},$$

is the cross ratio of a_0, a_1, a_2, a_3. Given any *ideal* tetrahedron (vertices at the boundary) in $\mathscr{H}^3$ and an ordering of its vertices, we can associate to it the cross ratio of its vertices which is a complex number in $\mathbf{C} - \{0, 1\}$. If we change the ordering of the vertices by an even permutation (so the orientation of the ideal tetrahedron is not changed) we obtain 3 possible answers for the cross ratio: z, $1 - 1/z$, and $1/(1 - z)$.

There are several definitions in the literature of the Bloch group $\mathscr{B}(F)$ of an infinite field F. They are isomorphic with each other modulo torsion of order dividing 6, and they are strictly isomorphic for F algebraically closed (see [6] and [14] for a more detailed discussion). The most widely used are those of Dupont and Sah [6] and of Suslin [19]. We shall use a definition equivalent to Dupont and Sah's, since it is the most appropriate for questions involving $PGL_2(F)$. Suslin's definition is appropriate for questions involving $GL_2(F)$ and gives a group which naturally embeds in ours with quotient of exponent at most 2. However, since torsion is irrelevant to our current considerations, the precise definition we choose is, in fact, unimportant.

Our $\mathscr{B}(F)$ is defined as follows. Let $\mathbf{Z}(\mathbf{C} - \{0, 1\})$ denote the free group generated by points in $\mathbf{C} - \{0, 1\}$. Define the map

$$\mu : \mathbf{Z}(\mathbf{C} - \{0, 1\}) \to \wedge^2_{\mathbf{Z}}(\mathbf{C}^*),$$

$$\sum_i n_i [z_i] \mapsto \sum_i n_i z_i \wedge (1 - z_i),$$

and let $\mathscr{A}(F) = \operatorname{Ker}(2\mu)$.

The Bloch group $\mathscr{B}(F)$ is the quotient of $\mathscr{A}(F)$ by all instances of the following "5-term relation"

$$[x] - [y] + [y/x] - \left[\frac{1 - x^{-1}}{1 - y^{-1}}\right] + \left[\frac{1 - y}{1 - x}\right] = 0,$$

where $x, y \in F - \{0, 1\}$. One checks easily that the map μ vanishes identically on this 5-term relation.

REMARK. The above 5-term relation is that of Suslin [19], which differs slightly from the following more widely used 5-term equation (first written down by Bloch-Wigner)

$$[x] - [y] + [y/x] - \left[\frac{1-y}{1-x}\right] + \left[\frac{1-y^{-1}}{1-x^{-1}}\right] = 0 .$$

These two different relations lead to mutually isomorphic $\mathscr{B}(\mathbf{C})$ via the map $[z] \mapsto [1/z]$. However, Suslin's 5-term relation seems more suitable for general F since it vanishes exactly under μ for an arbitrary F while the 5-term equation of Bloch-Wigner only vanishes up to 2-torsion [1]).

The geometric meaning of the 5-term equation is the following:

Let $a_0, \ldots, a_4 \in \mathbf{P}_1 = \mathscr{P}(\partial \mathscr{H}^3)$. Then the convex hull of these points in $\mathscr{H}^3$ can be decomposed into ideal tetrahedra in two ways, either as the union of the three ideal tetrahedra $\{a_1, a_2, a_3, a_4\}$, $\{a_0, a_1, a_3, a_4\}$ and $\{a_0, a_1, a_2, a_3\}$ or as the union of two ideal tetrahedra $\{a_0, a_2, a_3, a_4\}$ and $\{a_0, a_1, a_2, a_4\}$. Taking $(a_0, \ldots, a_4) = (0, \infty, 1, x, y)$, then the five terms in the above relation occur exactly as the five cross ratios of the corresponding ideal tetrahedra.

We discuss briefly how a hyperbolic 3-manifold determines an element in $\mathscr{B}(\mathbf{C})$. For more details, see [14].

It is straightforward for a closed hyperbolic 3-manifold M. Let $M = \mathscr{H}^3/\Gamma$, where Γ is a discrete subgroup of $PSL(2, \mathbf{C})$. The fundamental class of M in $H_3(M, \mathbf{Z})$ then gives rise to a class in $H_3(PSL_2(\mathbf{C}); \mathbf{Z})$ via the map

$$H_3(M, \mathbf{Z}) \cong H_3(\Gamma, \mathbf{Z}) \to H_3(PSL(2, \mathbf{C}), \mathbf{Z}) .$$

Fix a base point in $\mathbf{C} - \{0, 1\}$. Then via the natural action of $PSL(2, \mathbf{C})$ on $\mathbf{P}^1(\mathbf{C})$, any 4-tuple of elements in $PSL(2, \mathbf{C})$ gives a 4-tuple of elements in $\mathbf{C}$ which then gives rise to an element in $\mathbf{C}$ via the cross ratio. This induces a well defined homomorphism

$$H_3(PSL(2, \mathbf{C}), \mathbf{Z}) \to \mathscr{B}(\mathbf{C}) .$$

[1]) One could therefore give an alternate definition of the Bloch group by replacing 2μ by μ in the definition of $\mathscr{A}(F)$. This gives a group that lies between Suslin's Bloch group and the one defined here. Although this may seem more natural, the current definition is suggested by homological considerations. If F is algebraically closed the factor 2 makes no difference.

Although the cross ratio depends on the particular choice of the base point, the above homomorphism is independent of this choice (see [6, (4.10)]).

In the cusped case, we note that by [7], each cusped hyperbolic 3-manifold M has an ideal triangulation

$$M = \Delta_1 \cup \cdots \cup \Delta_n .$$

Let z_i be the cross ratio of any ordering of the vertices of Δ_i consistent with its orientation. The following relation was first discovered by W. Thurston (unpublished — see the remark in Zagier [21]; for a proof, see [14]):

$$\sum_i z_i \wedge (1 - z_i) = 0, \quad \text{in } \wedge^2_{\mathbf{Z}}(\mathbf{C}^*) .$$

This relation is independent of the particular choice of the ordering of the vertices of the Δ_i's.

From the definition of Bloch groups, it is thus clear that an ideal decomposition of a cusped hyperbolic 3-manifold determines an element $\beta(M) = \sum_i [z_i]$ in the Bloch group $\mathscr{B}(\mathbf{C})$. That such an element is independent of the choice of the ideal triangulation is proved in [14].

In fact, what we proved in [14] is stronger. It includes the following, which is what is needed for the purpose of this paper.

THEOREM 1.1. *Every hyperbolic 3-manifold M gives a well defined element $\beta(M)$ in the Bloch group $\mathscr{B}(\mathbf{C}) \otimes \mathbf{Q}$ which is the image of a well defined element $\beta_k(M)$ in $\mathscr{B}(k) \otimes \mathbf{Q}$ where k is the invariant trace field of M.* $\square$

2. BOREL'S THEOREM

The Bloch group of an infinite field F is closely related to the third algebraic K-group of F (denoted by $K_3(F)$). There exists a natural map

$$\mathscr{B}(F) \to K_3(F)$$

due to Bloch [2]. In particular, if F is a number field, then there is the isomorphism

$$\mathscr{B}(F) \otimes \mathbf{Q} \cong K_3(F) \otimes \mathbf{Q} .$$

For the precise relationship between the Bloch group and K_3 of an infinite field, see the work of Suslin [19].

In [3], Borel generalized the classical Dirichlet Unit Theorem in classical number theory. The generalization started with the observation that for a number field F, the unit group of the ring of integers $\mathscr{O}_F$ is non-other than the first algebraic K-group of $\mathcal{O}_F$, $K_1(\mathcal{O}_F)$. So the generalization is along the line of higher K-groups. For the exact statement, we refer to [3]. For the purpose of this paper, we state Borel's Theorem for K_3 of a number field in terms of the Bloch group. The way we state Borel's theorem here therefore incorporates a series of important works by Bloch and Suslin.

Define the *Bloch-Wigner* function $D_2 \colon \mathbf{C} - \{0, 1\} \to \mathbf{R}$ by (cf. [2])

$$D_2(z) = \operatorname{Im} \ln_2(z) + \log|z| \arg(1 - z), \quad z \in \mathbf{C} - \{0, 1\}$$

where $\ln_2(z)$ is the classical dilogarithm function. The hyperbolic volume of an ideal tetrahedron Δ with cross ratio z is equal to $D_2(z)$. It follows that D_2 satisfies the five-term functional equation given by the 5-term relation, and therefore D_2 induces a map

$$D_2 \colon \mathscr{B}(\mathbf{C}) \to \mathbf{R}\,,$$

by defining $D_2[z] = D_2(z)$.

Given a number field F, let r_1 and r_2 denote the number of real embeddings $F \subset \mathbf{R}$ and the number of pairs of conjugate complex embeddings $F \subset \mathbf{C}$ respectively. Let $\sigma_1, \ldots, \sigma_{r_1}, \sigma_{r_1+1}, \bar{\sigma}_{r_1+1}, \ldots, \sigma_{r_1+r_2}, \bar{\sigma}_{r_1+r_2}$ denote these embeddings. Given a number field F, then one has a map

$$\begin{aligned} c_2 : \quad & \mathscr{B}(F) \to \mathbf{R}^{r_2} \\ & \textstyle\sum_i (n_i[z_i]) \mapsto (\sum_i n_i D_2(\sigma_{r_1+1}(z_i)), \ldots, \sum_i n_i D_2(\sigma_{r_1+r_2}(z_i)))\,. \end{aligned}$$

BOREL'S THEOREM. *The kernel of* c_2 *is exactly the torsion subgroup of* $\mathscr{B}(F)$ *and the image of* c_2 *is a maximal lattice in* $\mathbf{R}^{r_2}$. *In particular, it follows that the rank of* $\mathscr{B}(F)$ *is* r_2. $\square$

This theorem has some useful consequences. Denote $\mathscr{B}(F)_{\mathbf{Q}} = \mathscr{B}(F) \otimes \mathbf{Q}$ for short. An immediate consequence of the theorem is that an inclusion of number fields $F \hookrightarrow E$ induces an injection $\mathscr{B}(F)/\mathit{Torsion} \to \mathscr{B}(E)/\mathit{Torsion}$ and hence an injection $\mathscr{B}(F)_{\mathbf{Q}} \to \mathscr{B}(E)_{\mathbf{Q}}$. Note that if $F \subset E$ is a finite Galois extension of fields with Galois group H then H acts on $\mathscr{B}(E)$.

PROPOSITION 2.1. *For any subfield* F *of the number field* E *identify* $\mathscr{B}(F)_{\mathbf{Q}}$ *with its image in* $\mathscr{B}(E)_{\mathbf{Q}}$. *Then if* F_1 *and* F_2 *are two subfields we have*

$$\mathscr{B}(F_1 \cap F_2)_{\mathbf{Q}} = \mathscr{B}(F_1)_{\mathbf{Q}} \cap \mathscr{B}(F_2)_{\mathbf{Q}}$$

and if E/F *is a Galois extension with group* H *then*

$$\mathcal{B}(F)_{\mathbf{Q}} = (\mathcal{B}(E)_{\mathbf{Q}})^H$$

(the fixed subgroup of $\mathcal{B}(E)_{\mathbf{Q}}$ *under* H*).*

Proof. This result appears to be known to experts but we could find no published proof, so we provide one. It clearly suffices to prove the results for the Bloch groups tensored with $\mathbf{R}$ rather than with $\mathbf{Q}$. We first prove the Galois property in the case that E is Galois over $\mathbf{Q}$. Let $G = \mathrm{Gal}(E/\mathbf{Q})$. If $\tau : E \to \mathbf{C}$ is our given embedding and $\delta \in G$ the restriction of complex conjugation for this embedding, then all complex embeddings have the form $\tau \circ \gamma$ with $\gamma \in G$ and the conjugate embedding to $\tau \circ \gamma$ is $\tau \circ \delta\gamma$. Consider the map $\mathcal{B}(E) \otimes \mathbf{R} \to \mathbf{R}G$ given on generators by

$$[z] \mapsto \sum_{\gamma \in G} D_2(\tau(\gamma(z)))\gamma \,.$$

By Borel's theorem this is injective with image exactly

$$\left\{ \sum_{g \in G} r_\gamma \gamma \mid r_\gamma = -r_{\delta\gamma} \text{ for all } \gamma \in G \right\}.$$

The G-action on $\mathcal{B}(E)$ corresponds to the action of G from the right on $\mathbf{R}G$. Thus, if we identify $\mathcal{B}(E) \otimes \mathbf{R}$ with its image in $\mathbf{R}G$, then $\mathcal{B}(E)^H \otimes \mathbf{R}$ is identified with the set of elements $\sum r_\gamma \gamma \in \mathbf{R}G$ satisfying $r_\gamma = -r_{\delta\gamma}$ and $r_\gamma = r_{\gamma\theta}$ for all $\gamma \in G$ and $\theta \in H$. That is, the function r_γ is constant on right cosets of H and for the cosets γH and $\delta\gamma H$ it is zero if they coincide and otherwise takes opposite values on each. Thus the rank r of $\mathcal{B}(E)^H \otimes \mathbf{R}$ is the half the number of cosets γH for which $\gamma^{-1}\delta\gamma \notin H$. Now, with $F = E^H$, the embedding $\tau \circ \gamma \mid F$ depends only on the coset γH and is real or complex according as its conjugation map $\gamma^{-1}\delta\gamma$ does or does not lie in H, so the above rank r is just $r_2(F)$. Since the image of $\mathcal{B}(F) \otimes \mathbf{R}$ lies in $\mathcal{B}(E)^H \otimes \mathbf{R}$ and has this rank, it must be all of $\mathcal{B}(E)^H \otimes \mathbf{R}$.

The case when E is not Galois over $\mathbf{Q}$ now follows easily by embedding E in a larger field which is Galois over $\mathbf{Q}$. Similarly, for the intersection formula, by replacing E by a larger field as necessary we may assume E is Galois over F_1 and F_2 with groups H_1 and H_2 say. Then, identifying $\mathcal{B}(F_i)_{\mathbf{Q}}$ with its image in $\mathcal{B}(E)_{\mathbf{Q}}$ we have $\mathcal{B}(F_1)_{\mathbf{Q}} \cap \mathcal{B}(F_2)_{\mathbf{Q}} = \mathcal{B}(E)_{\mathbf{Q}}^{H_1} \cap \mathcal{B}(E)_{\mathbf{Q}}^{H_2} = \mathcal{B}(E)_{\mathbf{Q}}^{\langle H_1, H_2 \rangle} = \mathcal{B}(E^{\langle H_1, H_2 \rangle})_{\mathbf{Q}} = \mathcal{B}(F_1 \cap F_2)_{\mathbf{Q}}$. $\square$

3. Proof and generalization of Theorem B

We shall give a proof of Theorem B based directly on Borel's theorem. A different proof can be given using Proposition 2.1 and its consequence Theorem 3.1.

We denote the complex conjugation in $\mathbf{C}$ by δ. Let F be a fixed non-real subfield of $\mathbf{C}$ that is stable under complex conjugation, i.e., $\delta(F) = F$. Assume F is a finite extension field of $\mathbf{Q}$. Let r_2 be as in Borel's Theorem. Then we may list all the complex (non-real) embeddings of F into $\mathbf{C}$ as $\tau_1, \delta\tau_1, \ldots, \tau_{r_2}, \delta\tau_{r_2}$. Let r_2' be the number of conjugate pairs that commutes with δ, i.e., $\tau_i\delta = \delta\tau_i$. Renumbering if necessary, we may assume $\tau_1, \ldots, \tau_l$ are the ones that commute with δ. Note that by our assumption on F, r_2' is at least one. The rest of the τ's won't commute with δ, therefore τ_i and $\tau_i\delta$, $i > r_2'$ will be in different conjugate pairs (we use here that F is non-real). So renumbering if necessary, we may assume

$$\tau_1, \ldots, \tau_{r_2'}, \tau_{r_2'+1}, \tau_{r_2'+1}\delta, \ldots, \tau_m, \tau_m\delta$$

gives exactly one representative from each conjugate pair of embeddings of F into $\mathbf{C}$, where $m = r_2' + (r_2 - r_2')/2$.

The complex conjugation on F induces an involution on $\mathscr{B}(F)$. Let $\mathscr{B}_+(F)$ and $\mathscr{B}_-(F)$ be the $\pm$ eigenspace of $\mathscr{B}(F)_{\mathbf{Q}} = \mathscr{B}(F) \otimes \mathbf{Q}$. By Borel's Theorem, $\mathscr{B}(F)_{\mathbf{Q}}$ has a $\mathbf{Q}$-basis $\alpha_1, \ldots, \alpha_{r_2}$. Let

$$u_i = \alpha_i - \delta\alpha_i, \quad v_i = \alpha_i + \delta\alpha_i, \; 1 \leqslant i \leqslant r_2 .$$

Then u_i's and v_i's span $\mathscr{B}_-(F)$ and $\mathscr{B}_+(F)$ respectively. Together, they span $\mathscr{B}(F)_{\mathbf{Q}}$. Hence by Borel's Theorem, we know that the matrix

$$\begin{pmatrix}
D_2(\tau_1(u_1)) & \cdots & D_2(\tau_m(u_1)) & D_2(\tau_{r_2'+1}\delta(u_1)) & \cdots & D_2(\tau_m\delta(u_1)) \\
\vdots & & \vdots & \vdots & & \vdots \\
D_2(\tau_{r_2}(u_{r_2})) & \cdots & D_2(\tau_m(u_{r_2})) & D_2(\tau_{r_2'+1}\delta(u_{r_2})) & \cdots & D_2(\tau_m\delta(u_{r_2})) \\
D_2(\tau_1(v_1)) & \cdots & D_2(\tau_m(v_1)) & D_2(\tau_{r_2'+1}\delta(v_1)) & \cdots & D_2(\tau_m\delta(v_1)) \\
\vdots & & \vdots & \vdots & & \vdots \\
D_2(\tau_{r_2}(v_{r_2})) & \cdots & D_2(\tau_m(v_{r_2})) & D_2(\tau_{r_2'+1}\delta(v_{r_2})) & \cdots & D_2(\tau_m\delta(v_{r_2}))
\end{pmatrix}$$

has rank r_2. Note that because the first r_2' embeddings commute with δ, the entries of the last r_2 rows of the first r_2 columns are all 0's. Also, it follows from the equation $\delta(u_i) = -\delta(u_i)$ and $\delta(v_i) = \delta(v_i)$, this matrix has the following block form

$$\begin{pmatrix} A_{r_2\times r_2'} & B_{r_2\times(r_2-r_2')/2} & -B_{r_2\times(r_2-r_2')/2} \\ 0 & C_{r_2\times(r_2-r_2')/2} & C_{r_2\times(r_2-r_2')/2} \end{pmatrix}$$

So the matrix A has to have rank r_2'. For the last $(r_2 - r_2')$-columns to have rank $r_2 - r_2'$, the matrices B and C must both have maximal possible rank, that is, $(r_2 - r_2')/2$. Since by Borel's Theorem,

$$\operatorname{rank} C = \operatorname{rank} \mathscr{B}_+(F),$$

and $\operatorname{rank} \mathscr{B}_+(F) + \operatorname{rank} \mathscr{B}_-(F) = r_2$, Theorem B follows. $\square$

We can also describe the situation when $F \subset \mathbf{C}$ is a number field that is not stable under conjugation. If $E \subset \mathbf{C}$ is any number field containing F with $E = \bar{E}$ then $\mathscr{B}_+(E)$ and $\mathscr{B}_-(E)$ are defined, so we can form

$$\mathscr{B}_+(F) := \mathscr{B}_+(E) \cap \mathscr{B}(F)_{\mathbf{Q}} \quad \text{and} \quad \mathscr{B}_-(F) := \mathscr{B}_-(E) \cap \mathscr{B}(F)_{\mathbf{Q}}.$$

These subgroups are independent of the choice of E, but in general they will not sum to $\mathscr{B}(F)_{\mathbf{Q}}$.

Denote $F_{\mathbf{R}} = F \cap \mathbf{R}$ and let $F' = F \cap \bar{F}$. Clearly F' contains $F_{\mathbf{R}}$, and $F_{\mathbf{R}}$ must be the fixed field of conjugation on F'. Thus either $F' = F_{\mathbf{R}}$ or F' is an imaginary quadratic extension of $F_{\mathbf{R}}$. Now F' is a field to which Theorem B applies, so $\mathscr{B}(F')_{\mathbf{Q}} = \mathscr{B}_+(F') \oplus \mathscr{B}_-(F')$, with the ranks of the summands given by Theorem B.

THEOREM 3.1. *$\mathscr{B}_-(F) = \mathscr{B}_-(F')$ and $\mathscr{B}_+(F) = \mathscr{B}_+(F') = \mathscr{B}(F_{\mathbf{R}})_{\mathbf{Q}}$.*

COROLLARY 3.2. *$\mathscr{B}_+(F)$ is trivial if and only if $F_{\mathbf{R}}$ is totally real.*

$\mathscr{B}_-(F)$ is trivial if and only if $F' = F_{\mathbf{R}}$.

$\mathscr{B}_-(F) = \mathscr{B}(F)_{\mathbf{Q}}$ if and only if $F = F'$ and $F_{\mathbf{R}}$ is totally real; that is either F is totally real or the embedding $F \hookrightarrow \mathbf{C}$ is a CM-embedding.

Proof of Theorem 3.1. We work in a Galois superfield E of F and identify Bloch groups with their images in $\mathscr{B}(E)_{\mathbf{Q}}$. Let $G = \operatorname{Gal}(E/\mathbf{Q})$, so $H = \operatorname{Gal}(E/F) \subset G$ is the subgroup which fixes F. We fix an embedding $E \subset \mathbf{C}$ extending the given embedding of F and denote complex conjugation for this embedding by δ. Then the subgroup $H_{\mathbf{R}}$ generated by H and δ is $\operatorname{Gal}(E/F_{\mathbf{R}})$, so it follows from Proposition 2.1 that $\mathscr{B}_+(F) = \mathscr{B}(E^H)_{\mathbf{Q}} \cap \mathscr{B}(E)_{\mathbf{Q}}^{\delta} = (\mathscr{B}(E)_{\mathbf{Q}}^{H})^{\delta} = \mathscr{B}(E)_{\mathbf{Q}}^{H_{\mathbf{R}}} = \mathscr{B}(F_{\mathbf{R}})_{\mathbf{Q}}$. Moreover, $\mathscr{B}_-(F)$ is fixed by both H and $\delta H \delta$ and hence by the group H' that they generate. But H' is the Galois group in E of $F \cap \delta(F) = F \cap \bar{F} = F'$. Thus $\mathscr{B}_-(F)$ is in

$\mathscr{B}(E)_{\mathbf{Q}}^{H'} = \mathscr{B}(F')_{\mathbf{Q}}$. We thus obtain inclusions $\mathscr{B}_{\pm}(F) \subset \mathscr{B}_{\pm}(F')$, and the reverse inclusions are trivial. □

Proof of Corollary 3.2. The first two statements follow immediately from Theorem 3.1 and Theorem B. For the third, note that $\mathscr{B}_{-}(F) \subset \mathscr{B}(F')_{\mathbf{Q}}$ and if $F' \neq F$ then F has strictly more complex embeddings than F' so $\mathscr{B}(F')_{\mathbf{Q}} \neq \mathscr{B}(F)_{\mathbf{Q}}$. Thus, to have $\mathscr{B}_{-}(F) = \mathscr{B}(F)_{\mathbf{Q}}$ we must have $F = F'$. The claim then follows directly from Theorem B. □

REMARK. We have pointed out at the beginning of sect. 2 that $\mathscr{B}(F)$ could have been replaced by $K_3(F)$ in all our discussions. The analog of Borel's theorem holds for $K_i(F)$ for all $i \equiv 3 \pmod 4$, so the results described above are also valid for these K-groups. When $1 < i \equiv 1 \pmod 4$ Borel's theorem gives a map $K_i(F) \to \mathbf{R}^{r_1+r_2}$ whose kernel is torsion and whose image is a lattice. The only change is that one obtains $r_1 + \frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ as the dimensions of the $+$ and $-$ eigenspaces in the analog of Theorem B, and Corollary 3.2 therefore also needs modification. We leave the details to the reader. The basic point is that if $E \subset \mathbf{C}$ is Galois over $\mathbf{Q}$ with group G and δ is its conjugation then $K_i(E) \otimes \mathbf{R}$ is G-equivariantly isomorphic to $\{\sum r_\gamma \gamma \in \mathbf{R}G \mid r_\gamma = (-1)^{(i-1)/2} r_{\delta\gamma}\}$ for $i > 1$ and odd.

4. MILNOR'S AND RAMAKRISHNAN'S CONJECTURES

Milnor [10] made the following conjecture motivated by the fact that $D_2(z)$ represents the volume of an ideal tetrahedron. For the significance of this conjecture in hyperbolic geometry and number theory, see [10], [11].

MILNOR'S CONJECTURE. *For each integer $N \geqslant 3$, the real numbers $D_2(e^{2\pi\sqrt{-1}j/N})$, with j relatively prime to N and $0 < j < N/2$, are linearly independent over the rationals.*

A field homomorphism $\tau: F \to K$ clearly induces a homomorphism on the Bloch groups $\mathscr{B}(F) \to \mathscr{B}(K)$ which, by abuse of notation, will again be denoted by τ.

Given a cyclotomic field $F = \mathbf{Q}(e^{2\pi\sqrt{-1}/N})$, the elements $[e^{2\pi ij/N}]$, with j relatively prime to N and $0 < j < N/2$, form a basis of the Bloch group $\mathscr{B}(F) \otimes \mathbf{Q}$ (see Bloch [2]). Hence Milnor's conjecture can be reformulated that $D_2: \mathscr{B}(F) \to \mathbf{R}$ given on generators by $[z] \mapsto D_2(z)$ is injective for a cyclotomic field F.

Note that for a general number field F the above map D_2 vanishes on $\mathscr{B}_+(F)$. By Corollary 3.2, the following is thus the strongest generalization of Milnor's conjecture that one might hope for.

CONJECTURE 4.1. *If* $F \subset \mathbf{C}$ *is a number field with* $F \cap \mathbf{R}$ *totally real then the map* $D_2 \colon \mathscr{B}(F) \to \mathbf{R}$ *given by* $[z] \mapsto D_2(z)$ *is injective.*

In the special case that F is Galois over $\mathbf{Q}$ the condition $F \cap \mathbf{R}$ totally real says that F is a CM-field. In this case we have the following proposition (cf. Prop. 7.2.5 of [16])

PROPOSITION 4.2. *Suppose that* F *is a Galois CM-field over* $\mathbf{Q}$. *If for one complex embedding* $\tau \colon F \hookrightarrow \mathbf{C}$, *the map* $D_2 \circ \tau$ *is injective (that is, Conjecture 4.1 holds), then it is for all complex embeddings.*

Proof. Let $\rho \colon F \hookrightarrow \mathbf{C}$ be another embedding of F. There exists $\gamma \in \mathrm{Gal}(F/\mathbf{Q})$ such that $\rho = \tau\gamma$. Let $\omega \in B(F) \otimes \mathbf{Q}$. If

$$D_2 \circ \rho(\omega) = D_2 \circ \tau(\gamma(\omega)) = 0 \,,$$

then by the injectivity of $D_2 \circ \tau$, $\gamma(\omega) = 0$. Since $\gamma \colon \mathscr{B}(F) \to \mathscr{B}(F)$ is clearly an isomorphism, it follows that $\omega = 0$ in $\mathscr{B}(F) \otimes \mathbf{Q}$. Hence, $D_2 \circ \rho$ is also injective. $\square$

The map $D_2 \colon \mathscr{B}(\mathbf{C}) \to \mathbf{R}$ is the imaginary part of the more general *Bloch* map

$$\rho \colon \mathscr{B}(\mathbf{C}) \to \mathbf{C}/\mathbf{Q}(2) \,,$$

where the notation $\mathbf{Q}(k)$ denotes the subgroup $(2\pi\sqrt{-1})^k\mathbf{Q}$ of $\mathbf{C}$. The definition of ρ is given as follows:

For $z \in \mathbf{C} - \{0, 1\}$, define

$$\rho(z) = \log z \wedge \log(1 - z) + 2\pi\sqrt{-1}$$
$$\wedge \frac{1}{2\pi\sqrt{-1}} (\ln_2(1 - z) - \ln_2(z) - \pi^2/6) \in \wedge^2_{\mathbf{Z}} \mathbf{C} \,.$$

See section 4 of [6] or [8] for the meaning of this map and for further details. The exact formulae given here is borrowed from [8]. This map obviously induces a map

$$\rho \colon \mathscr{A}(\mathbf{C}) \to \wedge^2_{\mathbf{Z}} \mathbf{C} \,.$$

It turns out that ρ vanishes on the 5-term relation, hence it induces a map

$$\rho \colon \mathscr{B}(\mathbf{C}) \to \wedge^2_{\mathbf{Z}} \mathbf{C} \,.$$

Finally, it follows from the fact that every element α of $\mathcal{B}(\mathbf{C})$ satisfies the relation $\mu(\alpha) = 0$ that the image of this last map lies in the kernel of the map

$$e : \wedge_{\mathbf{Z}}^2 \mathbf{C} \overset{\wedge^2 \exp}{\rightarrow} \wedge_{\mathbf{Z}}^2 \mathbf{C}^* .$$

The kernel of e is $\mathbf{C}/\mathbf{Q}(2)$. Hence this induces the Bloch map.

Ramakrishnan [16] generalized Milnor's conjecture in the following form[1]).

RAMAKRISHNAN'S CONJECTURE. *For every subfield* $F \overset{\sigma}{\hookrightarrow} \mathbf{C}$, *the map*

$$\mathcal{B}(F) \otimes \mathbf{Q} \overset{\sigma}{\rightarrow} \mathcal{B}(\mathbf{C}) \otimes \mathbf{Q} \rightarrow \mathbf{C}/\mathbf{Q}(2)$$

is injective.

The Bloch-Wigner function D_2 is the imaginary part of the Bloch map ρ, and it vanishes identically on $\mathcal{B}_+(k)$. On the other hand, it follows from a routine calculation that the real part of the Bloch map vanishes identically on $\mathcal{B}_-(k)$.

In particular, ρ just reduces to D_2 if $\mathcal{B}_-(F) = \mathcal{B}(F)_{\mathbf{Q}}$. By Corollary 3.2 we thus have

PROPOSITION 4.3. *If* $F \subset \mathbf{C}$ *is a CM-embedded field, then the Ramakrishnan Conjecture for this particular embedding of* F *is equivalent to Conjecture 4.1.* □

On the other hand

PROPOSITION 4.4. *The truth of the Ramakrishnan Conjecture for a field* $E = \bar{E} \subset \mathbf{C}$ *would imply Conjecture 4.1 for any subfield of* E.

Proof. Since the real and imaginary parts of ρ vanish on $\mathcal{B}_-(E)$ and $\mathcal{B}_+(E)$ respectively, the Ramakrishnan conjecture for E is equivalent to the conjecture that the kernel of the real part of ρ is exactly $\mathcal{B}_-(E)$ and the kernel of the imaginary part, that is $Ker(D_2)$, is exactly $\mathcal{B}_+(E)$. Thus D_2 would have zero kernel on $\mathcal{B}(F)_{\mathbf{Q}}$ for any subfield F of E satisfying $\mathcal{B}(F)_{\mathbf{Q}} \cap \mathcal{B}_+(E) = \{0\}$, which is equivalent to the condition of Conjecture 4.1 by Corollary 3.2. □

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[1]) Both Ramakrishnan's conjecture and Milnor's original conjecture are more general in that they apply to all the odd-degree higher K-groups. We refer to [16] for more details. Likewise, Conjecture 4.1 can be stated for higher K-groups in similar fashion.

5. Chern-Simons invariants and the regulator map

As we have seen from the above discussion, hyperbolic 3-manifolds and their volumes all have interesting K-theoretic interpretation. Another important invariant in the theory of hyperbolic 3-manifolds is the Chern-Simons invariant of [4] and [9]. What is its relation with K-theory? As discussed in sect. 1, any hyperbolic 3-manifold M represents a class $\beta(M)$ in $\mathscr{B}(\mathbf{C})$. For M compact, the next theorem follows from Theorem 1.11 of Dupont [5]. A proof for cusped M can be found in [14].

THEOREM 5.1. *The Chern-Simons invariant of* M *mod* $\mathbf{Q}(2)$ *is equal to the real part of the image of* $\beta(M)$ *under the regulator map*

$$\rho : \mathscr{B}(\mathbf{C}) \to \mathbf{C}/\mathbf{Q}(2)\,. \qquad \square$$

Now given a hyperbolic 3-manifold M with invariant trace field k and the associated embedding $\sigma : k \hookrightarrow \mathbf{C}$, the class $\beta(M) \in \mathscr{B}(\mathbf{C})$ associated to M is in the image of $\mathscr{B}(k)_{\mathbf{Q}} \xrightarrow{\sigma} \mathscr{B}(\mathbf{C})$ (Theorem 1.1). If σ is a CM-embedding, then it follows from Corollary 3.2 that the real part of ρ is trivial on $\mathscr{B}(F) \otimes \mathbf{Q}$. Theorem A in the introduction now follows immediately from Theorem 5.1. Similarly, as in the previous section, Corollary 3.2 implies that the irrationality conjecture for Chern Simons invariant of the Introduction would be implied by the Ramakrishnan Conjecture.

We have seen that the imaginary part of ρ is essentially the volume map while the real part of ρ can be called the Chern-Simons map. Reinterpreting the discussion of the previous section, this means that if $k = \bar{k} \subset \mathbf{C}$, then the volume map on $\mathscr{B}(k)$ factors through $\mathscr{B}_-(k)$ and Chern-Simons map on $\mathscr{B}(K)$ factors over $\mathscr{B}_+(k)$. By Theorem B we thus get bounds of $\frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$ on the number of rationally independent volumes resp. Chern-Simons invariants for manifolds having invariant trace field contained in our given k. Ramakrishnan's Conjecture says the image of ρ has $\mathbf{Q}$-rank r_2. This is equivalent to the conjecture that the $\mathbf{Q}$-ranks of the images of *vol*: $\mathscr{B}(k) \to \mathbf{R}$ and *CS*: $\mathscr{B}(k) \to \mathbf{R}/\mathbf{Q}$ are exactly $\frac{1}{2}(r_2 + r_2')$ and $\frac{1}{2}(r_2 - r_2')$.

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