5. The global form of the Stokes formula on C^1 manifolds

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as $\varphi(Q) = 0$ for any Q of the types (1)-(2) described above, and since φ is additive, it follows that $\psi(Q_i') = \varphi(Q_i)$ for any $i \in J$. In particular,

$$\left| \sum_{i \in J} \psi(Q_i') \right| = |\varphi(P)| \leqslant \left| \int_{\overset{\circ}{\Omega} \cap \partial P} u \right| + \left| \int_{\overset{\circ}{P} \cap b\Omega} u \right| + \left| \iint_{P \cap \Omega} f \right| .$$

By (4.2), the uniformly local $(n-1)$-integrability of u, the integrability of u on $b\Omega$ and the integrability of f on Ω, the right hand side of the above equality can be made arbitrarily small, provided $\sum_{i \in J} \mu_{n-1}(Q_i')$ is sufficiently small. However, since A' has Lebesgue measure zero in $\mathbf{R}^{n-1}$, this can be readily taken care of and this completes the proof of the theorem. $\square$

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally $(n-1)$-integrable form u is *uniformly* $(n-1)$-integrable only in a small neighborhood of $\mathscr{S}(u)$.

5. THE GLOBAL FORM OF THE STOKES FORMULA ON C^1 MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let M be a fixed, oriented, Hausdorff, differentiable manifold of class C^1, and real dimension n.

DEFINITION 5.1. *A subset Ω of M is called a C^1 domain if for any $a \in \Omega \setminus \overset{\circ}{\Omega}$, there exist an open neighborhood U of a in M and a C^1 diffeomorphism $f = (f_1, f_2, \ldots, f_n)$ of U onto an open neighborhood V of the origin in $\mathbf{R}^n$, such that*

$$U \cap \Omega = \{x \in U; f_n(x) \leqslant 0\} .$$

Clearly, the *border of the domain* $\Omega, b\Omega := \Omega \setminus \overset{\circ}{\Omega}$ is either the empty set or a $(n-1)$-dimensional C^1-submanifold of M assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any C^1 domain is also a Lipschitz domain in $\mathbf{R}^n$.

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of $\mathbf{R}^n$. In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust our previous definitions to the C^1 framework by carrying out the following simple modification. That is, whenever applicable, we shall replace "Lipschitz embedding" by "*C^1-embedding*", i.e. Lipschitz embeddings which are C^1 functions. Note that, in particular, the condition (1.1) is in this case equivalent with the continuity of the functions

$$\frac{\partial h(s,x)}{\partial x_i} : S \times \omega \to \Omega, \quad i = 1, 2, \dots, n-1 .$$

Assuming this modification, all the previously introduced notions become invariant to C^1 diffeomorphisms and, hence, meaningful on C^1 manifolds. More specifically, we make the following.

DEFINITION 5.2. *Let Ω be a C^1 domain of M. A $(n-1)$-form u is said to be absolutely continuous (uniformly $(n-1)$-locally integrable) on Ω if for any point $P \in \Omega$ there exists a local coordinate map $h: U \to \mathbf{R}^n$ of M with $P \in U$ such that $(h^{-1})^* u$ is absolutely continuous (uniformly $(n-1)$-locally integrable, respectively) on $h(U \cap \Omega)$.*

Let u and f be locally integrable forms on M, having degrees $(n-1)$ and n, respectively. Recall that $du = f$ on a open set Ω of M *in the distribution sense,* if for any $\varphi \in C_0^1(\Omega)$,

$$\int_M d\varphi \wedge u = - \int_M \varphi f .$$

THEOREM 5.3. *Let Ω be a C^1 domain of M, and u a $(n-1)$-form compactly supported in M. Assume that u is uniformly $(n-1)$-locally integrable and absolutely continuous on Ω, and that the singular set*

$$\mathcal{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \sup u$$

has $(n-1)$-dimensional Hausdorff measure zero.

If u is integrable on $b\Omega$ and du (taken in the sense of distribution theory) is integrable on Ω, then

$$\int_{b\Omega} u = \iint_{\mathring{\Omega}} du .$$

Proof. Using a smooth partition of unity and then working in local coordinates we can assume that $M = \mathbf{R}^n$. In this case, the conclusion is provided by Theorem 4.1. $\square$

Note that, here again it suffices to have the "uniform" part of the local $(n-1)$-integrability condition for u fulfilled only on a small neighborhood of $\mathscr{S}(u)$ (cf. also Remark 4.3).

DEFINITION 5.4. *A closed subset A of M is said to have an almost regular boundary if A coincides with the closure of its interior and if there exist a family $(S_i)_{i \in I}$ of C^1 submanifolds of M and a locally finite family $(C_i)_{i \in I}$ of compact subsets of M such that:*

(1) $C_i \subset S_i$, for any $i \in I$, and $\mathring{C}_i \cap \mathring{C}_j = \varnothing$, for all $i \neq j$ (the interiors are taken in S_i and in S_j, respectively);

(2) $C_i \cap C_j$ has $(n-1)$-dimensional Hausdorff measure zero for all $i \neq j$;

(3) $\partial A = \cup_{i \in I} C_i$.

Note that if A has an almost regular boundary, then

$$\Omega := \mathring{A} \cup \left(\bigcup_{i \in I} \mathring{C}_i \right)$$

(the interior of A is taken in M) is a C^1 domain with border $b\Omega = \cup_{i \in I} \mathring{C}_i$. If u is an integrally continuous $(n-1)$-form on M, it follows that u is integrable on each oriented submanifold $\mathring{C}_i$ (with the standard orientation induced by $\mathring{A}$). Since ∂C_i has zero measure in S_i, we can define

$$\int_{\partial A} u := \sum_{i \in I} \int_{\mathring{C}_i} u$$

whenever $A \cap \operatorname{supp} u$ is compact. Hence, without further proof, we can state the following.

THEOREM 5.5. *Let A be a subset of M with an almost regular boundary. If u is a $(n-1)$-form which is uniformly $(n-1)$-locally integrable on M, absolutely continuous on M, and for which $A \cap \operatorname{supp} u$ is compact, then u is integrable on ∂A, du is integrable on $\mathring{A}$ and*

$$\int_{\partial A} u = \iint_{\mathring{A}} du .$$