4. The global Stokes formula for simple Lipschitz domains in $\mathbf{R}^n$

A $(n-1)$-form u on $\mathbf{R}^n$ is said to be *uniformly locally $(n-1)$-integrable on* $\Omega \subseteq \mathbf{R}^n$ if it is locally $(n-1)$-integrable and, for any compact subset K of $\mathbf{R}^n$ and any $\varepsilon > 0$, there exists a positive $\delta = \delta(K, \varepsilon)$ such that

$$\left| \int_C u \right| < \varepsilon \tag{4.1}$$

whenever C is a $(n-1)$-dimensional Lipschitz submanifold C of $\mathbf{R}^n$ which is contained in $K \cap \Omega$ and has $\mu_{n-1}(C) < \delta$.

Examples include $(n-1)$-forms with locally bounded coefficients, or exhibiting isolated singularities of the type $\|x\|^{-\alpha}$, $\alpha < n-1$.

Let us recall the notion of simple Lipschitz domain introduced in the last part of Definition 1.1. The main result of this section is the following.

THEOREM 4.1. *Let Ω be a simple Lipschitz domain in $\mathbf{R}^n$, and let u be a compactly supported $(n-1)$-form in $\mathbf{R}^n$ which is uniformly $(n-1)$-locally integrable on $\mathbf{R}^n$. Assume that u is absolutely continuous on Ω and that the singular set*

$$\mathscr{S}(u) := (\bar{\Omega} \setminus \Omega) \cap \operatorname{supp} u$$

has $(n-1)$-dimensional Hausdorff measure zero.

Then, if u is integrable on $b\Omega$ and du (in the distribution sense) is integrable on Ω, we have

$$\int_{b\Omega} u = \iint_{\mathring{\Omega}} du .$$

To prove this theorem, we shall need an auxiliary lemma. Two Lipschitz domains Ω_1, Ω_2 in $\mathbf{R}^n$ will be called *almost transversal* if $\mu_{n-1}(b\Omega_1 \cap b\Omega_2) = 0$. Let Ω be a Lipschitz domain in $\mathbf{R}^n$ and let $\mathscr{R}$ stand for the collection of all rectangles of $\mathbf{R}^n$ which are almost transversal to Ω. Next, assume that u is a $(n-1)$-form compactly supported on $\mathbf{R}^n$, uniformly locally $(n-1)$-integrable on $\mathbf{R}^n$, and integrable on $b\Omega$. Also, let f be a locally integrable n-form on $\mathbf{R}^n$ and consider the complex-valued mapping φ defined on $\mathscr{R}$ by

$$\varphi(Q) := \int_{\mathring{Q} \cap b\Omega} U + \int_{\mathring{\Omega} \cap \partial Q} u - \iint_{Q \cap \Omega} f .$$

LEMMA 4.2. *Let* $\Omega, \mathscr{R}, u, f, \varphi$ *be as above and assume that* $\mathscr{S}(u) := (\bar{\Omega}\setminus\Omega) \cap \operatorname{supp} u$ *has Hausdorff* $(n-1)$*-dimensional measure zero. Then the following hold.*

(1) $\mathscr{R}$ *together with the usual subdivisions is a full rectangular system on* $\mathbf{R}^n$.

(2) If P *is a* $\mathscr{R}$*-paved set and* $(Q_i)_{i\in I}$ *is a subdivision of* P, *then*

$$\sum_{i\in I} \varphi(Q_i) = \int_{\overset{\circ}{P}\cap b\Omega} u + \int_{\overset{\circ}{\Omega}\cap\partial P} u - \iint_{P\cap\Omega} f .$$

In particular, φ *is additive.*

(3) The set $\mathscr{S}(u)$ *is* $(\varphi, 0)$*-negligible.*

Proof. For each $k = 1, 2, \ldots, n$, let A_k be the collection of all $c \in \mathbf{R}$ having the property that

$$\mu_{n-1}(\{x = (x_1, \ldots, x_n) \in b\Omega; x_k = c\}) > 0 .$$

Since $\lambda_n(b\Omega) = 0$, it follows by Fubini's theorem that A_k has Lebesgue measure zero in $\mathbf{R}$ for any k.

Consider now $Q, R_1, \ldots, R_m \in \mathscr{R}$ such that $R_\nu \subseteq Q$ for all ν. Let $(a_1, \ldots, a_n)$ be the origin of Q, and $(b_1, \ldots, b_n)$ the end-point of Q. Similarly, for each ν, $(a_1^\nu, \ldots, a_n^\nu)$ will stand for the origin of R_ν, whereas $(b_1^\nu, \ldots, b_n^\nu)$ will denote the end-point of R_ν. The almost transversality hypothesis implies that $a_k, b_k, a_k^\nu, b_k^\nu \in \mathbf{R}\setminus A_k$ for all ν, k.

Now, since $\lambda_1(A_k) = 0$, for any a priory given $\varepsilon > 0$, we can select a finite sequence of real numbers $x_{k,\alpha_k}^\nu \in \mathbf{R}\setminus A_k$, $\alpha_k = 0, \ldots, p_k$, such that

$$a_k = x_{k,0}^\nu < \cdots < x_{k,p_k}^\nu = b_k ,$$
$$|x_{k,\alpha_k-1}^\nu - x_{k,\alpha_k}^\nu| \leqslant \varepsilon n^{-1/2} ,$$

and, finally, so that a_k^ν and b_k^ν are among the numbers $\{x_{k,\alpha_k}^\nu\}_{\alpha_k}$. It is then easy to see that, for ε sufficiently small, the rectangles

$$Q_{(\alpha_1,\ldots,\alpha_n)} := \prod_{k=1}^{n} [x_{k,\alpha_k-1}, x_{k,\alpha_k}], \quad \text{with } 1 \leqslant \alpha_k \leqslant p_k ,$$

form an elementary subdivision of Q which contains a subdivision of R_ν for each $1 \leqslant \nu \leqslant m$. This completes the proof of (1).

Going further, (2) is immediate in the case in which the family $(Q_i)_{i\in I}$ comes from an elementary subdivision of a larger rectangle containing P. Thus, the general case then easily follows from this and (1).

Next we turn our attention to (3). Fix $Q \in \mathscr{R}, K$ a compact subset of $\Omega \setminus \mathscr{S}(u)$ and $\varepsilon > 0$. Since $\mathscr{S}(u)$ has $(n-1)$-dimensional Hausdorff measure zero, it is thus possible to select finitely many rectangles $R_1, \ldots, R_m \in \mathscr{R}$ which do not intersect K, their interiors cover $Q \cap \mathscr{S}(u)$, and such that

$$\sum_{\nu=1}^{m} \mu_{n-1}(\partial R_\nu) < \varepsilon .$$

Then $P := \cup_\nu (Q \cap R_\nu)$ is a $\mathscr{R}$-paved set contained in Q which does not intersect K and has the property that $\mu_{n-1}(\partial P) < \varepsilon$. Since $\mathscr{R}$ is full, we can find an elementary subdivision $(Q_i)_{i \in I}$ of Q and a subset J of I for which $P = \cup_{i \in J} Q_i$. In particular, we note that this implies $Q_i \cap \mathscr{S}(u) = \varnothing$ for $i \in I \setminus J$. Using (2), we can write

$$\sum_{i \in J} \varphi(Q_i) = \int_{\overset{\circ}{P} \cap b\Omega} u + \int_{\overset{\circ}{\Omega} \cap \partial P} u - \iint_P f .$$

Now, since u is integrable on $b\Omega$ and f is integrable on Ω, the first and the third terms from above can be made arbitrarily small by choosing K large enough. Furthermore, by taking ε sufficiently small and using the fact that u is uniformly locally $(n-1)$-integrable, the second term can also be made arbitrarily small. The proof of the lemma is therefore finished. $\square$

Proof of Theorem 4.1. Since in the conclusion of the theorem u intervenes only through its values on Ω, there is no loss of generality assuming that $u = 0$ on $\mathbf{R}^n \setminus \bar{\Omega}$, i.e. that $\operatorname{supp} u \subseteq \bar{\Omega}$ (note that this does not alter the hypotheses either). We set $f := du$ in $\overset{\circ}{\Omega}$, zero in $\mathbf{R}^n \setminus \overset{\circ}{\Omega}$, and adopt the notation introduced in Lemma 4.2. Clearly, it is enough to prove that $\varphi(Q) = 0$ for any $Q \in \mathscr{R}$. First, let us observe that from (the proof of) Theorem 1.3 this is immediate for rectangles of the following two types:

(1) $Q \subset \overset{\circ}{\Omega}$ or $u = 0$ on Q;

(2) after suitably permuting the coordinates in $\mathbf{R}^n$,

$$Q \cap \Omega = \{x = (x', x_n); x' \in Q' \quad \text{and} \quad a_n \leqslant x_n \leqslant \theta(x') < b_n\} ,$$

where $Q = Q' \times [a_n, b_n]$ and $\theta : \mathbf{R}^{n-1} \to (a_n, b_n)$ is a Lipschitz function. On the other hand, the compact set $\mathscr{S}(u)$ has zero μ_{n-1}-measure and, hence, by Lemma 4.2, is $(\varphi, 0)$-negligible. Consequently, using Theorem 3.4 with $s = t = 0$, it suffices to show that any point $a \in b\Omega$ has an open neighborhood $\mathscr{U}$ in $\mathbf{R}^n$ such that $\varphi(R) = 0$ for all rectangles $R \in \mathscr{R}$ included in $\mathscr{U}$ and containing a. By possibly relabeling the coordinates first, we can find an open rectangle U in $\mathbf{R}^n$ and a Lipschitz function $\theta: \mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$U \cap \Omega = U \cap \{x = (x', x_n); x_n \leqslant \theta(x')\} .$$

Now let $R = R' \times [a_n, b_n] \in \mathscr{R}$ be a fixed rectangle contained in U, where R' is a rectangle in $\mathbf{R}^{n-1}$ and $a_n, b_n \in \mathbf{R}$, $a_n < b_n$. Denote by $\mathscr{R}'$ the collection of all rectangles Q' from $\mathbf{R}^{n-1}$ which are contained in R', having $p(Q') \leqslant p(R') + 1$ and such that $Q' \times [a_n, b_n] \in \mathscr{R}$. Then, with the usual subdivisions, $(\mathscr{R}', \mathrm{div})$ becomes a rectangular system on R'.

Next, we introduce the mapping $\psi: \mathscr{R}' \to \mathbf{C}$ by setting

$$\psi(Q') := \varphi(Q' \times [a_n, b_n]), \quad Q' \in \mathscr{R}'$$

Thus, everything comes down to proving that ψ vanishes identically on $\mathscr{R}'$. Let us consider the following compact set in $\mathbf{R}^n$:

$$A' := R' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) .$$

If a rectangle $Q' \in \mathscr{R}'$ does not meet A', then the rectangle $Q' \times [a_n, b_n] \in \mathscr{R}$ is either of type (1) or (2) from above, so that, at any rate, $\psi(Q') = 0$.

Since φ is additive, so is ψ and, by the equivalence (1) $\Leftrightarrow$ (3) in Theorem 3.4 with $s = t = 0$, it suffices to prove that A' is $(\psi, 0)$-negligible. To this end, let $Q' \in \mathscr{R}'$ and let $(Q_i')_{i \in I}$ be a subdivision of Q' such that $\delta_i := \operatorname{diam}(Q_i') \leqslant \delta$, for all i, for some positive δ. We also introduce

$$J := \{i \in I; Q_i' \cap (\theta^{-1}(a_n) \cup \theta^{-1}(b_n)) \neq \varnothing\} .$$

For each $i \in J$ we have that at least one of the sets $Q_i' \cap \theta^{-1}(a_n)$, $Q_i' \cap \theta^{-1}(b_n)$ is empty provided δ is sufficiently small. Assuming that this is the case, we set

$$Q_i := Q_i' \times [a_n, a_n + \delta_i M]$$

if $Q_i' \cap \theta^{-1}(a_n) \neq \varnothing$, and

$$Q_i := Q_i' \times [b_n - \delta_i M, b_n] ,$$

if $Q_i' \cap \theta^{-1}(b_n) \neq \varnothing$. Here M stands for the (essential) supremum of $|\nabla \theta|$ on R'. Then $P := \cup_{i \in J} Q_i$ is a $\mathscr{R}$-paved set having

$$\mu_{n-1}(\partial P) \leqslant C \sum_{i \in J} \mu_{n-1}(Q_i') \tag{4.2}$$

for some positive constant C depending exclusively on θ and R'. Furthermore, as $\varphi(Q) = 0$ for any Q of the types (1)-(2) described above, and since φ is additive, it follows that $\psi(Q_i') = \varphi(Q_i)$ for any $i \in J$. In particular,

$$\left|\sum_{i \in J} \psi(Q_i')\right| = |\varphi(P)| \leqslant \left|\int_{\overset{\circ}{\Omega} \cap \partial P} u\right| + \left|\int_{\overset{\circ}{P} \cap b\Omega} u\right| + \left|\iint_{P \cap \Omega} f\right| .$$

By (4.2), the uniformly local $(n-1)$-integrability of u, the integrability of u on $b\Omega$ and the integrability of f on Ω, the right hand side of the above equality can be made arbitrarily small, provided $\sum_{i \in J} \mu_{n-1}(Q_i')$ is sufficiently small. However, since A' has Lebesgue measure zero in $\mathbf{R}^{n-1}$, this can be readily taken care of and this completes the proof of the theorem. □

REMARK 4.3. As an inspection of the proofs shows, Theorem 4.1 and Lemma 4.2 continue to hold in the case when the locally $(n-1)$-integrable form u is *uniformly* $(n-1)$-integrable only in a small neighborhood of $\mathscr{S}(u)$.

5. THE GLOBAL FORM OF THE STOKES FORMULA ON C^1 MANIFOLDS

In this section we shall present a coordinate free version of the main result of section 4. Throughout this section, we let M be a fixed, oriented, Hausdorff, differentiable manifold of class C^1, and real dimension n.

DEFINITION 5.1. *A subset Ω of M is called a C^1 domain if for any $a \in \Omega \setminus \overset{\circ}{\Omega}$, there exist an open neighborhood U of a in M and a C^1 diffeomorphism $f = (f_1, f_2, \ldots, f_n)$ of U onto an open neighborhood V of the origin in $\mathbf{R}^n$, such that*

$$U \cap \Omega = \{x \in U ; f_n(x) \leqslant 0\} .$$

Clearly, the *border of the domain* $\Omega, b\Omega := \Omega \setminus \overset{\circ}{\Omega}$ is either the empty set or a $(n-1)$-dimensional C^1-submanifold of M assumed with the standard induced orientation. Note that a simple application of the implicit function theorem shows that any C^1 domain is also a Lipschitz domain in $\mathbf{R}^n$.

It is not difficult to see that the class of Lipschitz domains described in Definition 1.1 is not invariant under the action of bi-Lipschitz diffeomorphisms of $\mathbf{R}^n$. In particular, Theorem 4.1 cannot be reformulated invariantly. To remedy this, for the rest of this section we shall slightly adjust