Introduction

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PLURIDIMENSIONAL ABSOLUTE CONTINUITY FOR DIFFERENTIAL FORMS AND THE STOKES FORMULA

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INTRODUCTION

The concept of absolute continuity for functions of one real variable (defined on an open set $\Omega \subseteq \mathbf{R}$) arises very naturally in connection with the problem of characterizing the largest class of functions $u : \Omega \to \mathbf{R}$ for which there exists $f \in L^1(\Omega, \mathrm{loc})$ such that the Leibnitz-Newton formula

$$u(b) - u(a) = \int_a^b f(x)\, dx \tag{0.1}$$

holds for any interval $[a, b] \subseteq \Omega$. Lebesgue's solution to this problem, i.e. that (0.1) holds if and only if u is (locally) absolutely continuous, establishes the most general (and natural) framework within which the Fundamental Theorem of Calculus works.

Over the years, the subject has continuously received a great deal of attention. In particular, considerable effort in the literature was devoted to generalizing this result in various respects; see for instance the monographs [Wh], [Fe3], [Sa], [BM], [La], [Ju1], [Zi], and the references therein.

One of the early recognized directions was to try to allow less regular integrands by generalizing the Lebesgue integral. For instance, the existence of derivatives which are not Lebesgue integrable was regarded as a shortcoming of Lebesgue's integral and not as a pathology of the functions under discussion. This point of view eventually led to the design of the Denjoy-Perron integral in the 1910's (cf. e.g. [Sa], Chapters VI, VII). For more recent developments along these lines we refer to the work of Harrison [Ha], Henstock [H], Kurzweil [Ku], Pfeffer [P1, 2, 3, 4], Mawhin [M1, 2].

Nonetheless, there are other natural ways to extend Lebesgue's theorem to higher dimensions and to extend its validity to more general integrands and domains while still using the usual Lebesgue integral. See, for instance, Whitney [Wh], Bochner [Bo], Shapiro [Sh1, 2, 3] among others. Another very important and influential work but having somewhat different aims is that of Federer [Fe1, 2, 3].

There are two major aspects of the corresponding problem in the pluridimensional setting.

(i) The local problem (i.e. the validity aspect). Describe the class of $(n-1)$-forms u on a domain $\Omega \subseteq \mathbf{R}^n$ for which there exists a n-form $f \in L^1(\Omega, \mathrm{loc})$ such that the following local Stokes formula holds:

$$\int_{\partial Q} u = \iint_Q f, \quad \text{for any rectangle } Q \subseteq \Omega . \tag{0.2}$$

(ii) The global problem (i.e. the invariant aspect). Find some minimal but also natural hypotheses on u so that the global Stokes formula

$$\int_{\partial\Omega} u = \iint_\Omega du \tag{0.3}$$

holds for a broad class of domains on C^1 manifolds.

The main goal of this work is to identify the essential analytical and geometrical assumptions needed to deal with *(i)* and *(ii)*. To treat the local problem we introduce the concept of absolute continuity for $(n-1)$-forms in $\mathbf{R}^n$. Being absolutely continuous turns out to be basically equivalent to the fact that the local Stokes formula holds true. It is important to point out that our definition is quite natural for it is homogeneous in n and reduces to the Lebesgue one when $n = 1$. Moreover, several alternative characterizations of this pluridimensional absolute continuity, much in the spirit of the one-dimensional results, can also be established.

Turning our attention to the global problem, let us first note that, due to the particular nature of the concept of rectifiability in the plane, the 2-dimensional case plays a special role in the literature. More concretely, many theorems initially stated in $\mathbf{R}^n$ can be further improved if $n = 2$ (see e.g. [P], [Lo], [JN]). However, since we shall try to formulate our main results with no artificial hypotheses and in as general a context as possible, we shall not attempt to single out this case in any way. Except for this particularity, our solution to the global problem is considerably more general than all the previously known forms of the Stokes theorem which go along the same coordinates. Moreover, both the validity context and its proof naturally reflect the scope of the theorem.

In addition to some necessary integrability assumptions, the differential form u satisfying (0.3) is assumed to be absolutely continuous and the singular set $S = (\bar{\Omega} \setminus \Omega) \cap \operatorname{supp} u$ is supposed to have $(n-1)$-dimensional Hausdorff measure zero, i.e. $\mu_{n-1}(S) = 0$. This should be compared, for instance, with Whitney's solution to the global problem in which the differential form u is assumed to be continuous and bounded outside of a singular set S satisfying certain geometric and measure theoretic conditions [Wh]. While these conditions do imply that $\mu_{n-1}(S) = 0$, the converse is, in general, false.

The key ingredient of the approach we present here is a localization method enabling us to pass from local, and even from infinitesimal, to global which we formalize and present in an axiomatic way. This is a synthesis as well as a significant extension of several basic procedures utilizing subdivision techniques. We refer to (the proofs of) Cousin's principle, Goursat's lemma, Pompeiu's removability theorem, etc.

The layout of the paper is as follows. The class of absolutely continuous differential forms is introduced and studied in § 1 and § 2. Among other things, here we show that for such forms the local Stokes formula is valid for arbitrary compact Lipschitz domains in place of rectangles. The localization technique alluded to earlier is devised in § 3. Global forms of the Stokes formula are obtained in § 4 for Lipschitz domains in $\mathbf{R}^n$ and, in invariant form, in § 5.

The last two sections are devoted to applications. The main results of § 6 give sufficient conditions under which the equalities $du = f$ and $\bar{\partial} u = f$ on $\Omega \setminus A$ (where A is a certain null set with a special structure) taken in the pointwise or in the distribution sense are actually valid on the entire domain Ω. In particular, for $f = 0$ and $u =$ function, we obtain very general removability criteria for holomorphic functions of several variables.

Finally, in §7, we record the Clifford algebra version of the results discussed in the previous sections: absolute continuity and the Leibnitz-Newton formula for Clifford-valued functions, removability criteria for monogenic functions, and the Pompeiu integral representation formula.

Before we begin the major part of this work, let us introduce some notation and definitions commonly used in the sequel. A *rectangle* in $\mathbf{R}^n$ will be any simplex Q of the form $Q := \prod_{k=1}^{n} [a_k, b_k]$, where $a_k, b_k \in \mathbf{R}$, $a_k < b_k$ for all k. The *eccentricity* of Q is given by

$$p(Q) := \sup_{1 \leqslant i, j \leqslant n} \frac{b_i - a_i}{b_j - a_j}.$$

Note that $p(Q) \geqslant 1$ and that Q is a *cube* precisely for $p(Q) = 1$. The lower left-most corner of Q, $(a_1, \ldots, a_n) \in \mathbf{R}^n$, will be called the *origin* of Q, whereas the upper right-most corner of Q, $(b_1, \ldots, b_n) \in \mathbf{R}^n$, the *end-point* of Q. The traces of the hyper-planes $\{x; x_k = a_k\}$ and $\{x; x_k = b_k\}$ on Q will be called the *faces* of Q. The collections of all rectangles contained in a subset Ω of $\mathbf{R}^n$ will be denoted by $\mathscr{R}(\Omega)$.

A *subdivision* of a rectangle Q is a finite collection of rectangles $(Q_i)_{i \in I}$ having mutually disjoint interiors and such that $\cup_{i \in I} Q_i = Q$. A subdivision of Q will be called *elementary* if its elements can be obtained as the Cartesian product of some fixed subdivisions of the factor intervals of Q.

More generally, the union $P = \cup_{i \in I} Q_i$ of finitely many rectangles $(Q_i)_{i \in I}$ with mutually disjoint interiors is called a *(compact) paved set,* and $(Q_i)_{i \in I}$ is said to be a *subdivision* of the paved set P.

The Euclidean space $\mathbf{R}^n$ is equipped with the usual metric $\|x\|^2 = \langle x, x \rangle = \sum_{i=1}^{n} x_i^2$, if $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. For $S \subset \mathbf{R}^n$, we set $\operatorname{diam}(S) := \sup\{\|x - y\|; x, y \in S\}$ and $\partial S := \bar{S} \setminus \overset{\circ}{S}$. Also, $\operatorname{comp}(S)$ will stand for the collection of all compact subsets of S. For $0 \leqslant r \leqslant n$, μ_r will denote the r-dimensional Hausdorff measure in $\mathbf{R}^n$, while λ_n will stand for the usual Lebesgue measure in $\mathbf{R}^n$. Finally, the $(n-1)$-dimensional and the n-dimensional Lebesgue integrals will be denoted by $\int$ and $\iint$, respectively.