2. The local collineation theorem

2. The local collineation theorem

In this section, we show that continuous local collineations of real or complex projective space are projective-linear or anti-projective-linear (Theorem 3). Our methods involve using Desargues' Theorem to extend to a global collineation and then applying the fundamental description of collineations over an arbitrary field (Proposition 1).

We let $\mathscr{L}^n_K$ denote the set of projective lines in projective n-space $\mathbf{P}^n_K$ over a field K. (We are interested here in the cases $K = \mathbf{R}$ or $\mathbf{C}$.) Note that $\mathscr{L}^n_K$ can be identified with the Grassmannian of 2-dimensional subspaces of K^{n+1}. A *collineation* on $\mathbf{P}^n_K$ is a bijective self-map $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ such that $f(L) \in \mathscr{L}^n_K$ for all $L \in \mathscr{L}^n_K$. Examples of collineations on $\mathbf{P}(K^{n+1})$ are provided by elements of the projective linear group $\mathrm{PGL}(n+1, K) = \mathrm{GL}(n+1, K)/(K \backslash \{0\})$. However, these are not the only collineations. We let the group $\mathrm{Gal}(K)$ of automorphisms of K (the Galois group of K over its prime field, $\mathbf{Z}_p$ or $\mathbf{Q}$) act on $\mathbf{P}^n_K$ by

$$g(z) = (gz_0 : \ldots : gz_n) \quad \text{for} \quad g \in \mathrm{Gal}(K), \quad z = (z_0 : \cdots : z_n) \in \mathbf{P}^n_K\,;$$

then elements of $\mathrm{Gal}(K)$ also give collineations on $\mathbf{P}^n_K$. The following well-known result (see [Ar, Theorem 2.26]) states that these examples provide all the collineations on $\mathbf{P}^n_K$:

Proposition 1. *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation, where $n \geqslant 2$ and K is an arbitrary field. Then there exist a unique $A \in \mathrm{PGL}(n+1, K)$ and a unique $g \in \mathrm{Gal}(K)$ such that $f = g \circ A$.*

We shall use of the following immediate consequence of Proposition 1:

Corollary 2. *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation, where $K = \mathbf{R}$ or $\mathbf{C}, n \geqslant 2$. Suppose f is continuous on a nonempty open subset of $\mathbf{P}^n_K$. If $K = \mathbf{R}$, then $f \in \mathrm{PGL}(n+1, \mathbf{R})$. If $K = \mathbf{C}$, then either f or $\bar{f}$ is in $\mathrm{PGL}(n+1, \mathbf{C})$.*

We let $\langle a_1, \ldots, a_m \rangle$ denote the projective linear subspace of $\mathbf{P}^n_K$ determined by the points $a_1, \ldots, a_m \in \mathbf{P}^n_K$. In particular, $\langle a, b \rangle$ is the projective line through a and b (for $a \neq b \in \mathbf{P}^n_K$). We also let a denote the one-point set $\langle a \rangle = \{a\}$. We now give a short proof of Proposition 1. First we need two well-known, elementary lemmas:

Lemma (a). *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation. If $a_1, \ldots, a_m$ are points in general position in $\mathbf{P}^n_K$, then $f(a_1), \ldots, f(a_m)$ are in general position and $f(\langle a_1, \ldots, a_m \rangle) = \langle f(a_1), \ldots, f(a_m) \rangle$.*

Proof. It suffices to consider $m \leqslant n + 1$. If $m = 1$ the conclusion is just the definition of a collineation. So let $2 \leqslant m \leqslant n + 1$ and assume by induction that the lemma has been verified for $m - 1$ points. We write $f(a) = \hat{a}$. Since $f(\langle a_1, \ldots, a_{m-1} \rangle) = \langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$ and f is injective, it follows that $\hat{a}_m \notin \langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$ and thus $\hat{a}_1, \ldots, \hat{a}_m$ are in general position. The second conclusion follows from the fact that $\langle \hat{a}_1, \ldots, \hat{a}_m \rangle$ is the union of lines $\langle \hat{a}_m, b \rangle$, where b runs through the points of $\langle \hat{a}_1, \ldots, \hat{a}_{m-1} \rangle$. $\square$

LEMMA (b). *Let $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation. If there exists a line $L \in \mathscr{L}^n_K$ such that $f|_L : L \to f(L)$ is projective-linear, then $f \in \mathrm{PGL}(n + 1, K)$.*

Proof. Let $\tilde{e}_j = (0, \ldots, \overset{j\text{-th}}{1}, \ldots, 0) \in K^{n+1}$, $0 \leqslant j \leqslant n$, $\tilde{\delta} = \widetilde{e_0} + \cdots + \widetilde{e_n}$, and let $e_0, \ldots, e_n, \delta$ be the corresponding points in $\mathbf{P}^n_K$. *Let* $f: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be as in the hypothesis; we can assume without loss of generality that $f|_{\langle e_0, e_1 \rangle}$ is projective-linear. By Lemma (a), the points $f(e_0), \ldots, f(e_n), f(\delta)$ are in general position. Choose representatives $\widetilde{f(e_0)}, \ldots, \widetilde{f(e_n)}, \widetilde{f(\delta)}$ in $K^{n+1} \backslash \{0\}$ of $f(e_0), \ldots, f(e_n), f(\delta)$ respectively. Let $\lambda_j \in K \backslash \{0\}$ $(0 \leqslant j \leqslant n)$ be given by $\sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$, and let $T \in GL(n + 1, K)$ be given by $T(\tilde{e}_j) = \lambda_j \widetilde{f(e_j)}$. Then $T(\tilde{\delta}) = \sum \lambda_j \widetilde{f(e_j)} = \widetilde{f(\delta)}$.

Let $\varphi = T^{-1} \circ f$. Thus the lemma is reduced to the following statement:

(A_n) *Let $\varphi: \mathbf{P}^n_K \to \mathbf{P}^n_K$ be a collineation such that $\varphi|_{\langle e_0, e_1 \rangle}$ is projective-linear, $\varphi(e_j) = e_j$ $(0 \leqslant j \leqslant n)$, and $\varphi(\delta) = \delta$. Then φ is the identity.*

We verify (A_n) by induction on n. For $n = 1$ the conclusion is immediate. So let $n \geqslant 2$ and assume (A_{n-1}). We write $\mathbf{P}^{n-1}_K = \langle e_0, \ldots, e_{n-1} \rangle$ and let $\delta' = (1 : \ldots : 1 : 0) \in \mathbf{P}^{n-1}_K$; thus $\langle e_n, \delta \rangle \cap \mathbf{P}^{n-1}_K = \{\delta'\}$. By Lemma (a), $\varphi(\mathbf{P}^{n-1}_K) = \mathbf{P}^{n-1}_K$ and thus $\varphi(\delta') = \delta'$. Hence by ($\mathrm{A}_{n-1}$), φ is the identity on $\mathbf{P}^{n-1}_K$. If a line $L \in \mathscr{L}^n_K$ contains a point $b \notin \mathbf{P}^{n-1}_K$ such that $\varphi(b) = b$, then $\varphi(L) = L$, since L must contain another fixed point of φ in $\mathbf{P}^{n-1}_K$. Let $a \in \langle e_0, e_n \rangle$, $a \neq e_0$, be arbitrary. Since $\{a\} = \langle a, \delta \rangle \cap \langle e_0, e_n \rangle$ and the points δ, e_n are fixed by φ, it follows that $\varphi(\langle a, \delta \rangle) = \langle a, \delta \rangle$ and $\varphi(\langle e_0, e_n \rangle) = \langle e_0, e_n \rangle$ and thus $\varphi(a) = a$. Finally, let $x \in \mathbf{P}^n_K \backslash \langle e_0, e_n \rangle$ be arbitrary. Since $\{x\} = \langle a, x \rangle \cap \langle e_n, x \rangle$, where a is as above and φ fixes a, e_n, it follows as before that $\varphi(x) = x$. $\square$

Proof of Proposition 1. Consider the usual embeddings $\mathbf{P}^1_K \subset \mathbf{P}^2_K \subset \mathbf{P}^n_K$. By Lemma (a), $f(\mathbf{P}^2_K)$ is a projective 2-plane. Hence there exists a projective linear map $T: f(\mathbf{P}^2_K) \to \mathbf{P}^2_K$ such that the map $f' = T \circ f|_{\mathbf{P}^2_K} : \mathbf{P}^2_K \to \mathbf{P}^2_K$ leaves the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$ and $(1:1:1)$ fixed. Then, for each $a \in K$, we can write $f'(1:a:0) = (1:\hat{a}:0)$, where $\hat{a} \in K$. We observe that the map $a \mapsto \hat{a}$ is an element of $\mathrm{Gal}(K)$. This follows from the fact that if $a, b \in K$, then $a - b$ and a/b can be constructed from the following "projective straightedge" constructions:

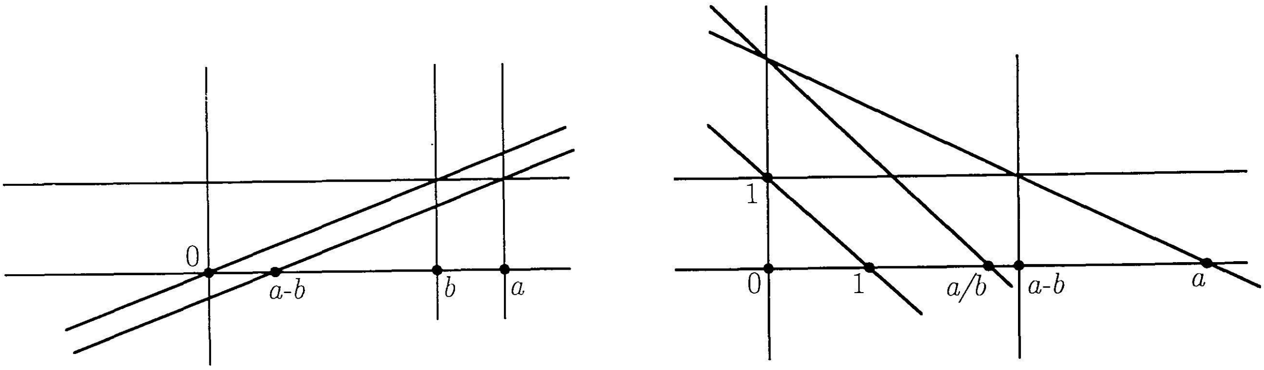

FIGURE 0

(Figure 0 shows the affine plane $K^2 \subset \mathbf{P}^2_K$.) Let $g \in \mathrm{Gal}(K)$ with $g(a) = \hat{a}$. Then $f' \circ g^{-1}|_{\mathbf{P}^1_K}$ is the identity map, and it follows that the map $f \circ g^{-1}|_{\mathbf{P}^1_K} : \mathbf{P}^1_K \to f(\mathbf{P}^1_K)$ is projective-linear. Therefore by Lemma (b), $f \circ g^{-1} = A' \in \mathrm{PGL}(n+1, K)$, and thus $f = A' \circ g = g \circ A$, where $A = g^{-1}A'g \in \mathrm{PGL}(n+1, K)$. $\square$

For a subset $U \subset \mathbf{P}^n_K$, we write

$$\mathscr{L}(U) = \{L \in \mathscr{L}^n_K : L \cap U \neq \emptyset\}\,.$$

We give the projective spaces $\mathbf{P}^n_{\mathbf{R}}, \mathbf{P}^n_{\mathbf{C}}$ and the Grassmannians $\mathscr{L}^n_{\mathbf{R}}, \mathscr{L}^n_{\mathbf{C}}$ the usual metric topologies. The main result of this section gives a condition for a local collineation to be projective-linear:

THEOREM 3. *Let U be a connected open set in $\mathbf{P}^n_K (n \geqslant 2)$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $\mathscr{L}_0$ be an open subset of $\mathscr{L}(U)$ such that $\bigcup \mathscr{L}_0 \supset U$. Suppose that $f: U \to \mathbf{P}^n_K$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for all $L \in \mathscr{L}_0$. Then there exists $A \in \mathrm{PGL}(n+1, K)$ such that*

(i) $f = A|_U$, *if* $K = \mathbf{R}$,

(ii) $f = A|_U$ *or* $\bar{f} = A|_U$, *if* $K = \mathbf{C}$.

The case $K = \mathbf{R}$ of Theorem 3 follows easily from Prenowitz's theorem [Pr, Theorem V], which provides a much stronger result for $n = 2$. (We include an elementary proof of the case $K = \mathbf{R}$ below.)

We begin by proving the following weaker form of Theorem 3:

LEMMA 4. *Let U be an open set in $\mathbf{P}^n_K (n \geqslant 2)$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $f: U \to \mathbf{P}^n_K$ be a continuous injective map. If $f(L \cap U)$ is contained in a projective line for all $L \in \mathscr{L}(U)$, then the conclusion of Theorem 3 holds.*

Proof. Let $f: U \to \mathbf{P}^n_K$ be as in the statement of the lemma, and let $f(U) = \hat{U}$. We write $\hat{a} = f(a)$ for $a \in U$. Note that if three points a_1, a_2, a_3 of U are not collinear, then $\hat{a}_1, \hat{a}_2, \hat{a}_3$ are not collinear, since otherwise the sets $f(\langle a_1, a_2 \rangle \cap U)$ and $f(\langle a_1, a_3 \rangle \cap U)$ would both be neighborhoods of a_1 in the line $\langle \hat{a}_1, \hat{a}_2 \rangle$ and hence f would not be injective. We also observe that if $L = \langle a, b \rangle$, where a, b are distinct points of U, then by hypothesis, $f(L \cap U) \subset \langle \hat{a}, \hat{b} \rangle$, and in fact we have $f(L \cap U) = \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$. To verify this equality, let $\chi \in \langle \hat{a}, \hat{b} \rangle \cap \hat{U}$ be arbitrary and write $\chi = \hat{x}$, where $x \in U$. Since $\hat{a}, \hat{b}, \hat{x}$ are collinear, it follows from the above that x, a, b are collinear and thus $x \in L$.

We first consider the case $n = 2$. Choose a connected open set $U_0 \subset U$. Let $x \in \mathbf{P}^2_K$. We want to define $\hat{x} = \tilde{f}(x)$. Choose $a, b \in U_0$ such that a, b, x are not collinear. Let $\hat{L}_a, \hat{L}_b \in \mathscr{L}(\hat{U})$ be given by $f(\langle a, x \rangle \cap U) = \hat{L}_a \cap \hat{U}$, $f(\langle b, x \rangle \cap U) = \hat{L}_b \cap \hat{U}$. We define $\hat{x}(a, b) \in \mathbf{P}^2_K$ by

$$\hat{L}_a \cap \hat{L}_b = \hat{x}(a, b) .$$

(Note that $\hat{L}_a \neq \hat{L}_b$ since $\langle a, x \rangle \neq \langle b, x \rangle$ and f is injective.)

We observe that if $a' \in \langle a, x \rangle \cap U_0$, $b' \in \langle b, x \rangle \cap U_0$ with $a' \neq a$, $b' \neq b$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}, \hat{b}' \rangle .$$

In particular if $x \in U$, then

$$\hat{x}(a, b) = \langle \hat{a}, \hat{x} \rangle \cap \langle \hat{b}, \hat{x} \rangle = \hat{x} .$$

STEP 1. *$\hat{x}(a, b)$ is independent of the choice of $a, b \in U_0$.*

We can assume by the above that $x \notin U$. Let $a \in U_0$ and let $b_0, b_1 \in U_0 \setminus \langle a, x \rangle$ be arbitrary. It suffices to show that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$.

We first consider the case $K = \mathbf{C}$. Let C be a real curve from b_0 to b_1 in $U_0 \backslash \langle a, x \rangle$. Let $\varepsilon > 0$, and suppose that b_2, b_3 are points in C such that $\mathrm{dist}(b_2, b_3) < \varepsilon$ (with respect to some metric on $\mathbf{P}^2_{\mathbf{C}}$ defining the usual topology). Choose $a', a'' \in \langle a, x \rangle \cap U_0$ with a, a', a'' distinct. Then let

$$b_3', b_3'', b_2', c, b_2''$$

be constructed (in the above order) as in Figure 1 below.

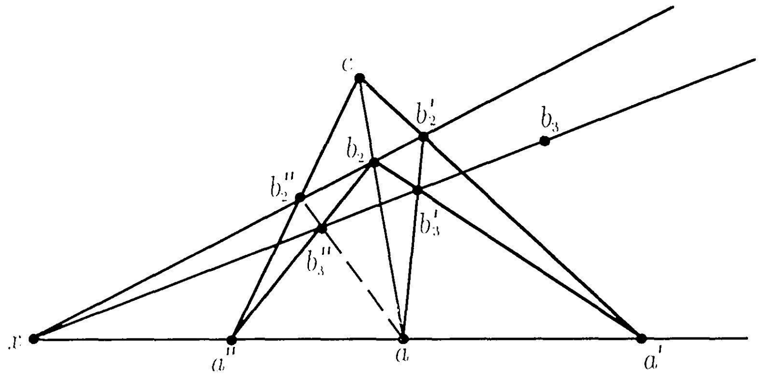

FIGURE 1

We claim that a, b_2'', b_3'' are collinear: Let $b_3^* = \langle a, b_2'' \rangle \cap \langle a'', b_2 \rangle$; to verify the claim, we must show that $b_3^* = b_3''$. By Desargues' Theorem [Co, 2.32; see Fig. 4.4a on p. 39] b_3', b_3^*, x are collinear and thus

$$b_3^* \in \langle b_3', x \rangle \cap \langle a'', b_2 \rangle = b_3'' ,$$

as desired.

We note that if $b_3 = b_2$, then

$$b_2 = b_3' = b_3'' = b_2' = c = b_2'' .$$

Since C is compact, it follows that we can choose ε small enough so that all the labeled points in Figure 1 except x lie in U_0 whenever b_2, b_3 are points of C with $\mathrm{dist}(b_2, b_3) < \varepsilon$. Again by Desargues' Theorem, $\langle \hat{a}, \hat{a}' \rangle$, $\langle \hat{b}_2, \hat{b}_2' \rangle$ and $\langle \hat{b}_3', \hat{b}_3'' \rangle$ are coincident. Thus

$$\begin{aligned} \hat{x}(a, b_2) = \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_2, \hat{b}_2' \rangle &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3', \hat{b}_3'' \rangle \\ &= \langle \hat{a}, \hat{a}' \rangle \cap \langle \hat{b}_3, \hat{b}_3' \rangle = \hat{x}(a, b_3) . \end{aligned}$$

It follows that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, which completes Step 1 for the case $K = \mathbf{C}$.

We now suppose that $K = \mathbf{R}$. (The proof must be modified for the case $K = \mathbf{R}$, since $U_0 \backslash \langle a, x \rangle$ may not be connected.) We may assume without loss of generality that the line segment

$$C \stackrel{\text{def}}{=} \{tb_0 + (1-t)b_1 : 0 \leqslant t \leqslant 1\}$$

is contained in U_0. If $C \cap \langle a, x\rangle = \emptyset$, then we conclude that $\hat{x}(a, b_0) = \hat{x}(a, b_1)$, by the proof for the case $K = \mathbf{C}$ above. On the other hand, if $C \cap \langle a, x\rangle = b'$, then

$$\hat{x}(b_0, a) = \hat{x}(b_0, b') = \hat{x}(b_0, b_1) = \hat{x}(b', b_1) = \hat{x}(a, b_1),$$

which completes Step 1 for the case $K = \mathbf{R}$.

We now write $\hat{x} = \hat{x}(a, b) = \tilde{f}(x)$ for all $x \in \mathbf{P}^2_K$.

STEP 2. $\tilde{f}$ *is a collineation.*

Let x, y, z be collinear. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. Choose collinear points $a, b, c \in U_0 \setminus \langle x, y\rangle$. Let a', b', c' be as in Figure 2 below. We note that if $a = b = c$, then $a' = b' = c' = a$. Thus we can choose distinct collinear $a, b, c \in U_0 \setminus \langle x, y\rangle$ such that a', b', c' are in U_0. By moving the line $\langle a, b\rangle$ slightly if necessary, we can assume further that $x, y, z \notin \langle a, b\rangle$, and hence a', b', c' are distinct. By Pappas' Theorem (see for example [Co, 4.41 and Fig. 4.4a]), a', b', c' are collinear. It further follows from the above that no four of the nine labeled points in Figure 2 are collinear. By the collinearity of f on U, the points $\hat{a}, \hat{b}, \hat{c}$ are collinear and distinct, and the same is true for $\hat{a}', \hat{b}', \hat{c}'$; furthermore, no four of the points $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}'$ are collinear. Hence $\hat{x}, \hat{y}, \hat{z}$ are distinct, and thus $\tilde{f}$ is injective. Applying Pappas' Theorem again (with $a, b, c, x, y, z, a', b', c'$ replaced by $\hat{a}, \hat{b}, \hat{c}, \hat{a}', \hat{b}', \hat{c}', \hat{x}, \hat{y}, \hat{z}$, respectively), we conclude that $\hat{x}, \hat{y}, \hat{z}$ are collinear.

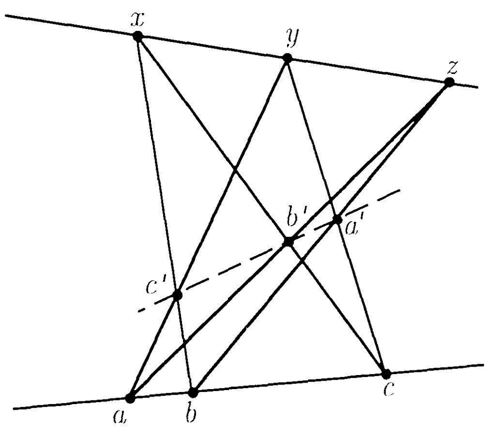

FIGURE 2

Finally, to show that $\tilde{f}$ is surjective, let $\chi \in \mathbf{P}^2_K$ be arbitrary. Choose points $\alpha, \alpha', \beta, \beta' \in \hat{U}_0 = f(U_0)$ such that $\chi = \langle \alpha, \alpha'\rangle \cap \langle \beta, \beta'\rangle$. The points $\alpha, \alpha', \beta, \beta'$ are the respective images of points $a, a', b, b' \in U_0$. If we set $x = \langle a, a'\rangle \cap \langle b, b'\rangle$, then $\chi = \hat{x}$.

Hence $\tilde{f}$ is a collineation. The case $n = 2$ then follows from Corollary 2.

STEP 3. *The proof for* $n > 2$.

Let $n > 2$. We easily see that f takes 2-planes in U to 2-planes in $\hat{U}$. Let $L \in \mathscr{L}(U)$ be arbitrary. By applying the case $n = 2$ to a projective 2-plane containing L, we see that $f|_{L \cap U}: L \cap U \to \hat{L} \cap \hat{U}$ is either projective-linear or anti-projective-linear. If $f|_{L \cap U}$ is anti-projective-linear for one L, it must be anti-projective-linear for all L (by the case $n = 2$), so by replacing f with $\bar{f}$ if necessary, we can assume that $f|_{L \cap U}$ is projective-linear for all $L \in \mathscr{L}(U)$. Now fix $a \in U$. For $x \in \mathbf{P}^n_K$, define $\hat{x} = T(x)$ where $T: \langle a, x \rangle \to \langle \hat{a}, \hat{x} \rangle$ is the projective-linear transformation extending $f|_{\langle a, x \rangle \cap U}$. By applying the case $n = 2$ to the plane determined by a, a', x (for an arbitrary point $a' \notin \langle a, x \rangle$), we see that $\hat{x}$ is independent of a. Thus we can define $\tilde{f}(x) = \hat{x}$. If x, y, z are collinear and $a \notin \langle x, y \rangle$, then the case $n = 2$ applied to the plane determined by a, x, y implies that $\hat{x}, \hat{y}, \hat{z}$ are collinear. The injectivity of $\tilde{f}$ similarly follows from the case $n = 2$. To show surjectivity, let $\chi \in \mathbf{P}^n_K$ be arbitrary, and choose a point $\alpha \in \langle \hat{a}, \chi \rangle \cap \hat{U} \setminus \{\hat{a}\}$. Then α is the image of a point $a' \in U$ and $\tilde{f}(\langle a, a' \rangle) = \langle \hat{a}, \alpha \rangle$. Hence $\chi \in \langle \hat{a}, \alpha \rangle \subset$ image $\tilde{f}$.

Thus $\tilde{f}$ is a collineation. The conclusion of the lemma follows as before from Corollary 2. □

DEFINITION. A subset U of $\mathbf{P}^n_{\mathbf{R}}$ or $\mathbf{P}^n_{\mathbf{C}}$ is said to be *projectively convex* if $L \cap U$ is connected for all projective lines $L \in \mathscr{L}(U)$. (Note that if $U \subset \mathbf{R}^n \subset \mathbf{P}^n_{\mathbf{R}}$, then U is projectively convex if and only if U is convex.)

We use the following lemma to complete the proof of Theorem 3:

LEMMA 5. *Let U be a projectively convex, open set in $\mathbf{P}^n_K$, where K denotes either $\mathbf{R}$ or $\mathbf{C}$, and let $\mathscr{L}_0$ be an open subset of $\mathscr{L}(U)$ such that $\bigcup \mathscr{L}_0 \supset U$. Suppose that $f: U \to \mathbf{P}^n_K$ is a continuous injective map such that $f(L \cap U)$ is contained in a projective line for each $L \in \mathscr{L}_0$. Then $f(L \cap U)$ is contained in a projective line for every $L \in \mathscr{L}(U)$.*

Proof. We again write $\hat{p} = f(p)$, for $p \in U$. Let $L \in \mathscr{L}(U)$ be arbitrary, and let $x \in L \cap U$. Since $L \cap U$ is connected, it suffices to show that there is a neighborhood $V \subset U$ of x such that $\hat{x}, \hat{y}, \hat{z}$ are collinear whenever $y, z \in L \cap V$. Choose a line $L_x \in \mathscr{L}_0$ containing x. We can assume that $L_x \neq L$, since otherwise we are done. Choose $w \in L_x \cap U$, $w \neq x$. Next choose a neighborhood $V \subset U$ of x such that $\langle y, w \rangle \in \mathscr{L}_0$ for all $y \in V$.

Let $y, z \in L \cap V$. We must show that $\hat{x}, \hat{y}, \hat{z}$ are collinear. We can assume that x, y, z are distinct points. Choose $v \in L \cap V$ distinct from x, y, z (see Figure 3). Since $\langle v, w \rangle \in \mathscr{L}_0$, we can choose $a \in L_x \setminus \{x, w\}$ sufficiently close to w so that the line $L_a = \langle v, a \rangle \in \mathscr{L}_0$. Let $b = \langle y, w \rangle \cap L_a$, $c = \langle z, w \rangle \cap L_a$. By choosing a close enough to w, we can assume further that $a, b, c \in U$ and the six lines

$$\langle x, b \rangle, \ \langle x, c \rangle, \ \langle y, a \rangle, \ \langle y, c \rangle, \ \langle z, a \rangle, \ \langle z, b \rangle$$

are in $\mathscr{L}_0$. Let a', b', c' be as in Figure 3. Since all the points and lines of Figure 3 lie in a plane, we can use Desargues' Theorem to conclude that v, a', b', c' are collinear. Write $L' = \langle v, c' \rangle$; thus $a', b' \in L'$. Since a', b', c' (as well as b, c) converge to w as $a \to w$, by choosing a sufficiently close to w we can assume also that $a', b', c' \in U$ and $L' \in \mathscr{L}_0$. Since all the labeled points in Figure 3 lie in U and all the lines in Figure 3 except L are in $\mathscr{L}_0$, we conclude that the f-images of the points in Figure 3 lie in the plane determined by the image lines $\widehat{L_a}$ and $\widehat{L_x}$. We now apply Pappas' Theorem to the image to conclude (as in Step 2 of the proof of Lemma 4) that $\hat{x}, \hat{y}, \hat{z}$ are collinear. $\square$

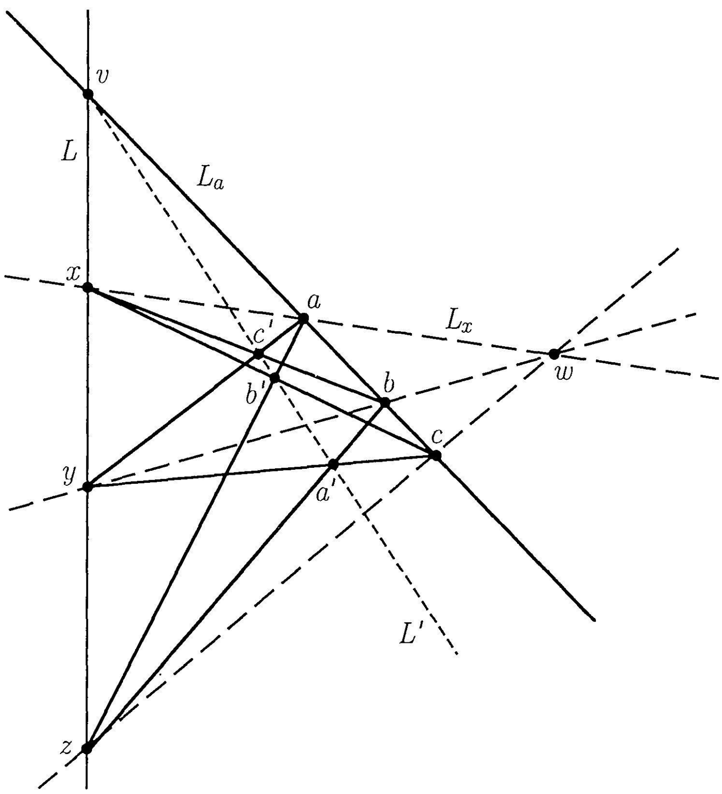

FIGURE 3

Proof of Theorem 3. Choose a sequence $\{U_1, U_2, \ldots\}$ of projectively convex, open subsets of U such that $U = \bigcup_{j=1}^{\infty} U_j$ and $U_1 \cup \cdots \cup U_j$ is connected for each $j \geqslant 1$. If $K = \mathbf{R}$, let $G = \mathrm{PGL}(n+1, \mathbf{R})$; if $K = \mathbf{C}$, let $G = \{e, \tau\} \cdot \mathrm{PGL}(n+1, \mathbf{C})$, where $\tau: \mathbf{P}^n_{\mathbf{C}} \to \mathbf{P}^n_{\mathbf{C}}$ is given by $\tau(z) = \bar{z}$ and e is the identity map. By Lemmas 5 and 4 applied to the restrictions $f|_{U_j}$, there are transformations $A_j \in G$ such that $f|_{U_j} = A_j|_{U_j}$. Since an element of G is uniquely determined by its values on a nonempty open subset of $\mathbf{P}^n_K$ and $(U_1 \cup \cdots \cup U_j) \cap U_{j+1} \neq \emptyset$, it follows by induction that $A_j = A_1$ for all j. Hence $f = A_1|_U$. □

3. The Poincaré-Tanaka and Chern-Ji theorems

The Segre family $\mathcal{M}_{B_n}$ mentioned in the introduction has the projective analogue

$$\mathcal{M}^n_K = \{(z, w) \in \mathbf{P}^n_K \times \mathbf{P}^n_K : \sum_{j=0}^{n} z_j w_j = 0\}.$$

(In fact $\mathcal{M}^n_K$ is a compactification of $\mathcal{M}_{B_n}$; see the proof of Corollary 8.) We let $\pi_i: \mathbf{P}^n_K \times \mathbf{P}^n_K \to \mathbf{P}^n_K$ denote the projection to the i-th factor, for $i = 1, 2$. The main result of this section is the following generalization of the Chern-Ji theorem [CJ, Theorem 2]; our generalization says that a pair of local homeomorphisms of $\mathbf{P}^n_K$ ($K = \mathbf{R}$ or $\mathbf{C}$) mapping $\mathcal{M}^n_K$ into itself must be projective-linear, or possibly anti-projective-linear (if $K = \mathbf{C}$):

Theorem 6. *Let $(a^1, a^2) \in \mathcal{M}^n_K$, where $K = \mathbf{R}$ or $\mathbf{C}$, $n \geq 2$. Let U_1, U_2 be open sets in $\mathbf{P}^n_K$ containing a^1, a^2 respectively, and let V_i be the connected component of $\pi_i(\mathcal{M}^n_K \cap U_1 \times U_2)$ containing a_i, for $i = 1, 2$. If $f_i: U_i \to \mathbf{P}^n_K$ $(i = 1, 2)$ are continuous injective maps such that*

$$(f_1 \times f_2)(\mathcal{M}^n_K \cap U_1 \times U_2) \subset \mathcal{M}^n_K,$$

then there exists $A \in \mathrm{PGL}(n+1, K)$ such that

(i) *$f_1 = A$ on V_1 and $f_2 = {}^tA^{-1}$ on V_2, if $K = \mathbf{R}$,*

(ii) *either* (i) *holds or $\bar{f}_1 = A$ on V_1 and $\bar{f}_2 = {}^tA^{-1}$ on V_2, if $K = \mathbf{C}$.*

Remark. If the sets $\pi_i(\mathcal{M}^n_K \cap U_1 \times U_2)$ are connected, then $V_i = \pi_i(\mathcal{M}^n_K \cap U_1 \times U_2)$ and we have $\mathcal{M}^n_K \cap U_1 \times U_2 = \mathcal{M}^n_K \cap V_1 \times V_2$. In fact, if we assume that only one of the projections $\pi_1(\mathcal{M}^n_K \cap U_1 \times U_2)$ is connected, then by the uniqueness of A it follows that the conclusion of Theorem 6 holds with $V_i = \pi_i(\mathcal{M}^n_K \cap U_1 \times U_2)$, for $i = 1, 2$.