

1. Introduction

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **41 (1995)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.04.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

SYNTHETIC PROJECTIVE GEOMETRY AND POINCARÉ'S THEOREM ON AUTOMORPHISMS OF THE BALL

by Bernard SHIFFMAN¹⁾

1. INTRODUCTION

Let B_n denote the unit ball in $\mathbf{C}^n$. In 1907, Poincaré [Po] showed that any nonconstant holomorphic map f from a neighborhood $U \subset \mathbf{C}^2$ of a point $z_0 \in \partial B_2$ into $\mathbf{C}^2$ which maps $U \cap \partial B_2$ into ∂B_2 must be the restriction of an element of the Möbius group of automorphisms of B_2 . This result was generalized to n variables by Tanaka [Ta] and was given new proofs by Pelles [Pe], Alexander [Al], Rudin [Ru], and others, and recently by Chern and Ji [CJ]. Chern and Ji considered the “Segre family” of ∂B_n ,

$$\mathcal{M}_{B_n} = \{(z, w) \in \mathbf{C}^n \times \mathbf{C}^n : \sum_{j=1}^n z_j w_j = 1\},$$

and showed that if $(z_0, w_0) \in \mathcal{M}_{B_n}$ and if f, g are nondegenerate holomorphic maps from neighborhoods U, V of z_0, w_0 , respectively, into $\mathbf{C}^n$ such that $f \times g$ maps $\mathcal{M}_{B_n} \cap (U \times V)$ into $\mathcal{M}_{B_n}$, then both f and g are restrictions of elements of the Möbius group [CJ, Theorem 2]. The Poincaré-Tanaka theorem follows easily from this result by considering the point $(z_0, \bar{z}_0) \in \mathcal{M}_{B_n}$ and taking $g(w) = \overline{f(w)}$ (see §3). The method of Segre families was also used in this context by S. Webster [We], who showed that local holomorphic maps of nondegenerate real-algebraic hypersurfaces in $\mathbf{C}^n$ are algebraic.

In this paper, we show how the methods of Desarguesian projective geometry provide an elementary proof of the Chern-Ji theorem. Since our methods are “synthetic”, we do not use any differential geometry, and apart from some complex analysis used in the proof of the Poincaré-Tanaka theorem, our proofs use only linear algebra and point-set topology and are self-contained (except for the omission of the proofs of the fundamental theorems

¹⁾ Research partially supported by National Science Foundation Grant No. DMS-9204037.

of Desargues and Pappus, which can be found in most texts on plane projective geometry, e.g. [Co]). In fact we show (Theorem 6) that the Chern-Ji theorem extends to the case of continuous f, g (where the conclusion holds either for f, g or for their conjugates). Our method is based on the principle that a continuous local self-map of real or complex projective space is projective-linear or anti-projective-linear (in the complex case) if it maps each line in a sufficiently large family $\mathcal{L}_0$ of lines into a line. For the case of the real projective plane $\mathbf{P}_R^2$, this principle was stated by Blaschke and his co-workers in the 1920s (see [BB, p. 91]) when $\mathcal{L}_0$ is a “4-web”; i.e., $\mathcal{L}_0$ consists of four pairwise transversal families of lines, each covering the domain of the map. A complete proof of this fact was given in 1935 by W. Prenowitz [Pr] (see also [Re]). We give a simple proof of this principle for the case where $\mathcal{L}_0$ is an open set in the Grassmannian of projective lines in real or complex projective n -space (Theorem 3).

Various other results on extending local collineations have appeared in the literature. For example, E. Cartan [Ca] showed that a self-map of the boundary of the 2-ball B_2 that takes any linear section in ∂B_2 into a complex line must be either projective-linear or anti-projective-linear. Radó (see [Ra]) observed that a collineation on any subset of a projective plane $\mathbf{P}_K^2$ (over any field K) that contains three generic lines and a generic point extends to a collineation of the entire projective plane. Mok and Yeung [MY, pp. 257-258] showed that local holomorphic collineations are projective-linear; a generalization of this result to biholomorphisms of complex manifolds preserving the geodesics of a projective connection was recently given by Molzon and Mortensen [MM, Theorem 9.1]. Some applications of Blaschke’s theory of webs to algebraic geometry can be found in Chern-Griffiths [CG]. (For an overview of the theory of webs, see [Go].) Also, the Poincaré-Tanaka theorem was generalized by Alexander and Rudin to the case where f is a holomorphic map from a domain $\Omega \subset B_n$ whose boundary contains an open subset of ∂B_n onto a similar domain. Alexander [Al] showed that if f has a C^∞ extension to $\bar{\Omega}$ that maps $\bar{\Omega} \cap \partial B_n$ into ∂B_n , then f extends to an automorphism of B_n ; Rudin [Ru, Theorem 15.3.4] replaced Alexander’s hypothesis by a much weaker condition that is satisfied, for example, when f has a continuous extension to $\bar{\Omega}$ mapping $\bar{\Omega} \cap \partial B_n$ into ∂B_n . (For discussions of related results, see [Fo, pp. 325-326] and [Ru, §15.3].)

Acknowledgements. I would like to thank Valery Alexeev, Shiing-Shen Chern, Shanyu Ji and Sid Webster for their helpful suggestions and references to the literature.