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where, as in (3.3), ∂_i is the derivation of $\mathscr{S}$ extending the functional $\lambda \mapsto \lambda(H_i)$. We have a perfect pairing

$$\mathscr{D} \otimes \mathscr{S} \to \mathbf{R}$$

given by $(D, f) = (Df)(0)$. Since the pairing is perfect, something in degree ν must pair nontrivially with Π. Since an irreducible W-module can only pair nontrivially with its dual, and the self-dual character ε occurs with multiplicity one in $\mathscr{D}^\nu$, afforded by $\partial_1 \cdots \partial_\nu$, we must have $\partial_1 \cdots \partial_\nu \Pi \neq 0$, so $c(\Pi) \neq 0$.

Observe that $\partial_1 \cdots \partial_\nu$ is analogous to the fundamental class of G/T, and the pairing is essentially that between homology and cohomology. We further remark that in fact all irreducible representations of W are defined over the rational numbers, hence they are all self dual. This is a consequence of Springer's cohomological construction of W-modules [Sp].

Returning again to our task, we now inductively assume that $c: \mathscr{H}^k \to H^{2k}(G/T)$ is injective for $k \leqslant \nu$, and let $V = \mathscr{H}^{k-1} \cap \ker c$. Note that V is preserved by W since c is W-equivariant. The sign character does not occur in $\mathscr{H}^{k-1}$, so there is a positive root α whose corresponding reflection s_α does not act by $-I$ on V. Decompose $V = V_+ \oplus V_-$ according to the eigenspaces of s_α. If $V \neq 0$ then $V_+ \neq 0$, so take $f \in V_+$. Now $c(\alpha f) = c(\alpha) c(f) = 0$, and αf is in degree k, so we must have $\alpha f \in \mathscr{I}$ by the induction hypothesis. Let $h_1, \ldots, h_{|W|}$ be a basis of $\mathscr{H}$ with $h_1, \ldots, h_r$ s_α-skew and the rest s_α invariant. By Chevalley's theorem (3.2), we can write $\alpha f = \sum h_i \sigma_i$ with σ_i W-invariant of positive degree. Since αf is s_α-skew, the sum only goes up to r. Now for $i \leqslant r$, the polynomial h_i must vanish on $\ker \alpha$, hence can be written $h_i = \alpha h_i'$ for some $h_i' \in \mathscr{S}$. But then $f = \sum_{i=1}^r h_i' \sigma_i \in \mathscr{I}$. Since f is supposed to be harmonic, we must have $f = 0$. Hence c is injective on $\mathscr{H}$, and the proof of Borel's theorem is complete. □

6. The cohomology of a Lie group

We now have all the ingredients for our proof. Consider the map $\psi: G/T \times T \to G$ given by $\psi(gT, t) = gtg^{-1}$. The Weyl group W acts on T by conjugation and on G/T by $w \cdot gT = gn^{-1}T$, where $w = nT$. Hence W acts on $H(G/T \times T) = H(G/T) \otimes H(T)$. Since $\psi(gn^{-1}T, wtw^{-1}) = \psi(gT, t)$, it follows that the induced map ψ^* on cohomology maps $H(G)$ to $[H(G/T) \otimes H(T)]^W$. Though we prefer to have it in this form, the latter group could be thought of as the cohomology of the quotient of $G/T \times T$

by the action of W, and this quotient has a natural interpretation. As in the introduction, let M be the set of pairs (g, T') where T' is a maximal torus in G containing $g \in G$. The map $G/T \times_W T \to M$ sending $(gT, t) \bmod (W)$ to (gtg^{-1}, gTg^{-1}) is a diffeomorphism.

PROPOSITION (6.1). *The map induced by* ψ *on cohomology is an isomorphism of graded rings*

$$\psi^*: H(G) \xrightarrow{\cong} [H(G/T) \otimes H(T)]^W .$$

Proof. We compute the derivative $(d\psi)_{(gT,t)}$ at a point $(gT, t) \in G/T \times T$. For each point $gT \in G/T$, we identify $\mathfrak{m}$ with the tangent space $T_{gT}(G/T)$ by letting $X \in \mathfrak{m}$ correspond to the initial tangent vector X_{gT} of the path $s \mapsto g(\exp sX)T$ in G/T. Similarly, an element $X \in \mathfrak{g}$ (resp. $H \in \mathfrak{t}$) corresponds to a tangent vector $X_g \in T_g(G)$ (resp. $H_t \in T_t(T)$, for $t \in T$). Then

$$\begin{aligned}
(d\psi)_{gT,t}(X_{gT}, 0) &= \frac{d}{ds} g(\exp sX)t(\exp - sX)g^{-1}\big|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[\exp sAd(gt^{-1})X]\,[\exp - sAd(g)X]\big|_{s=0} \\
&= \frac{d}{ds} gtg^{-1}[X + sAd(g)\,(Ad(t^{-1}) - 1)X + O(s^2)]\big|_{s=0} \\
&= [Ad(g)\,(Ad(t^{-1}))X]_{gtg^{-1}} .
\end{aligned}$$

Similarly, we find, for $H \in \mathfrak{t}$, that

$$(d\psi)_{gT,t}(0, H_t) = [Ad(g)H]_{gtg^{-1}} .$$

Hence, under the identifications, $(d\psi)_{(gT,t)}$ is the map

$$(Ad(t^{-1}) - I)_{\mathfrak{m}} \oplus I: \mathfrak{m} \oplus \mathfrak{t} \to \mathfrak{m} \oplus \mathfrak{t} = \mathfrak{g} .$$

Here the subscript $\mathfrak{m}$ means we view $Ad(t^{-1}) - I$ as a map from $\mathfrak{m}$ to itself. Now G being compact and connected, we must have $\det Ad(t) = 1$, so

$$\det (d\psi)_{(gT,t)} = \det (I - Ad(t))_{\mathfrak{m}} .$$

(Actually, $\mathfrak{m}$ is always even-dimensional as we have seen, so there is no need to reverse the subtraction).

We compute the degree of ψ by finding a regular value. Let t_0 be a generic element in T, as in (2.3). Consider $\psi^{-1}(t_0) = \{(gT, t): gtg^{-1} = t_0\}$. It turns out that any two elements of T conjugate in G must be conjugate by an element of W. (In $U(n)$, two diagonal matrices with the same set of eigenvalues must be conjugate by a permutation matrix.) It follows easily then that

$$\psi^{-1}(t_0) = \{(wT, wt_0w^{-1}) : w \in W\} .$$

We next show that ψ preserves orientation at each point in $\psi^{-1}(t_0)$. The eigenvalues of $Ad(t_0)$ in $\mathfrak{m}$ are complex conjugate pairs $z, \bar{z}$, where $|z| = 1, z \neq 1$. Hence $|1 - z||1 - \bar{z}| = 2(1 - Re(z)) > 0$, so $\det(I - Ad(t_0))_{\mathfrak{m}} > 0$.

At this point we know the degree of ψ is $\deg \psi = |W| \neq 0$. By Poincaré duality, any smooth map between compact manifolds of the same dimension is injective on cohomology as soon as it has nonzero degree. Hence $\psi^* : H(G) \to [H(G/T) \times H(T)]^W$ is injective. We finish the proof of (6.1) by showing that both sides have the same dimension.

For this we use, three times, the following basic principle. Let K be a compact group (here K will be G, T or W). Let dk be the left invariant Haar measure on K with total mass one. Let V be a finite dimensional real vector space, and $\rho : K \to GL(V)$ a continuous group homomorphism. Then the space V^K of vectors fixed by all $\rho(k)$, $k \in K$, has dimension

$$\dim V^K = \int_K \operatorname{trace} \rho(k)\, dk .$$

To compute this integral over G, we must exploit further the computation of $d\psi$. Let $\omega_G, \omega_T, \omega_{G/T}$ be the unique invariant (under left translations by G, T, and G respectively) differential forms of top degree whose integral over the respective manifold is one. The the pull-back formula for integration gives

$$\int_G f\omega_G = \frac{1}{\deg \psi} \int_{G/T \times T} f \circ \psi(gT, t)\, |\det(d\psi)_{(gT,t)}|\, \omega_{G/T} \wedge \omega_T ,$$

where the determinant is computed with respect to bases spanning parallelograms of unit volume with respect to the appropriate forms. Taking f to be invariant under conjugation by G, we arrive at the famous Weyl integration formula:

$$\int_G f\omega_G = \frac{1}{|W|} \int_T f(t) \det(I - Ad(t))_{\mathfrak{m}}\, \omega_T .$$

Expand the function $t \mapsto \det(I - Ad(t))_{\mathfrak{m}}$ in a sum of characters of T: $n_0\chi_0 + n_1\chi_1 + \cdots + n_k\chi_k$. Here χ_0 is the trivial character of T, appearing n_0 times, and for $i > 0$ each χ_i is a nontrivial character appearing n_i times. Taking for f the constant function equal to one, and applying the basic principle of invariants to T, we find $n_0 = |W|$.

Taking for f the function $f(g) = \det(I + Ad(g))$, i.e., the trace of $Ad(g)$ acting on $\Lambda\mathfrak{g}$, we find, using the Cartan-de Rham isomorphism (4.3), that

$$\dim H(G) = \dim(\Lambda\mathfrak{g})^G = \int_G \det(I + Ad(g))\omega_G$$

$$= \frac{1}{|W|}\int_T \det(I + Ad(t))\det(I - Ad(t))_{\mathfrak{m}}\omega_T$$

$$= \frac{2^{\dim T}}{|W|}\int_T \det(I - Ad(t^2))_{\mathfrak{m}}\omega_T .$$

Now the squaring map on T is surjective, so the square of a nontrivial character of T is still nontrivial. Hence the trivial character again appears with multiplicity $|W|$ in the expansion of $\det(I - Ad(t^2))_{\mathfrak{m}}$. This multiplicity is the value of the integral, so $\dim H(G) = 2^{\dim T} = 2^l$.

On the other hand, we saw in (5.3) that the trace of $w \in W$ acting on $H(G/T)$ is $|W|$ if $w = 1$, zero otherwise. Applying the invariance formula one more time, we find that $\dim [H(G/T) \otimes H(T)]^W = 2^l$ as well, completing the proof of (6.1). □

We now have the main result

(6.2) THEOREM. *The cohomology ring* $H(G)$ *with real coefficients is a bigraded exterior algebra with generators in bi-degrees* $(2m_i, 1)$, *for* $1 \leqslant i \leqslant l$.

Proof. By (6.1) and (5.4), we have

$$H(G) \simeq [H(G/T) \otimes H(T)]^W \simeq [\mathscr{H}_{(2)} \otimes \Lambda]^W ,$$

and by (3.8), the latter space is an exterior algebra with generators in degrees $(2m_i, 1)$, for $1 \leqslant i \leqslant l$. □

Moreover, from the multiplicity formula (3.8), the dimensions of the bi-graded pieces are given in terms of the exponents as follows

(6.3) COROLLARY. *For each* $q \geqslant 0$, *we have*

$$\sum_{n=0}^{\dim G} \dim [H^{n-q}(G/T) \otimes H^q(T)]^W u^n = u^q s_q(u^{2m_1}, \ldots, u^{2m_l}) .$$

(6.4) We give two interpretations of the bigrading. First, we follow [L] and consider the spectral sequence of the fibration $G \to G/T$, which has E_2-term

$$E_2^{pq} = H^p(G/T) \otimes H^q(T) ,$$

and converges to $H(G)$. This spectral sequence does not degenerate at E_2, but it has a spectral subsequence which does degenerate, and still converges to $H(G)$.

To see this we again consider the Weyl group action. More precisely, N acts by automorphisms of the fibration $G \to G/T$, which in turn induce automorphisms of each term in the spectral sequence, commuting with the differentials. On $E_2^{pq} = H^p(G/T) \otimes H^q(T)$, the action of N factors through W and is the same as that considered above. Thus we have representations of W on the spaces E_2^{pq}, hence on each E_r^{pq} for $r \geqslant 2$.

For each p, q, r we decompose $E_r^{pq} = (E_r^{pq})^W \oplus (E_r^{pq})_W$, where the subscript W indicates a W-stable complement to the invariants. Each of the latter two spaces is a spectral subsequence, and since E_∞^{pq} is a subquotient of $H^{p+q}(G)$ and N acts trivially on $H(G)$ (because G is connected), we must have $(E_\infty^{pq})_W = 0$. On the other hand, $(E_\infty^{pq})^W$ is a subquotient of $(E_2^{pq})^W = [H^p(G/T) \otimes H^q(T)]^W$, so we have

$$\begin{aligned} 2^l = \dim H(G) &= \sum_{p,q} \dim (E_\infty^{pq})^W \leqslant \sum_{p,q} \dim (E_2^{pq})^W \\ &= \sum_q \dim [H(G/T) \otimes \Lambda^q]^W = 2^l . \end{aligned}$$

It follows that $\dim (E_\infty^{pq})^W = \dim (E_2^{pq})^W$ for all pq, so the spectral subsequence of W-invariants degenerates at $(E_2)^W$, and (6.1) is proved again.

(6.5) The significance of the bigrading on $H(G)$ can be seen in yet another way, inspired by [GHV]. We consider, for a fixed integer $k \neq 1$, the k^{th}-power maps $x \mapsto x^k$, denoted p_k and P_k, on T and G, respectively. It is shown in [GHV] that the Lefschetz number of P_k equals that of p_k, namely $(1-k)^l$. Let $H^n(G)_q$ be the k^q-eigenspace of P_k^* acting on $H^n(G)$. It is further shown in [GHV] that $\sum_n \dim H^n(G)_q = \binom{l}{q}$. We can refine this by computing each $\dim H^n(G)_q$ separately. Consider the commutative diagram

$$\begin{array}{ccc} H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W . \\ p_k^* \downarrow & & \downarrow 1 \otimes p_k^* \\ H(G) & \xrightarrow{\psi^*} & [H(G/T) \otimes H(T)]^W . \end{array}$$

Since p_k^* acts by k^q on $H^q(T)$, (6.1) implies that $H^n(G)_q \simeq [H^{n-q}(G/T) \otimes H^q(T)]^W$, and (6.3) gives the dimension of the latter space.

(6.6) This last interpretation of the bigrading shows that it is natural in the following sense. Suppose $f: K \to G$ is a homomorphism between two compact connected Lie groups. Since f commutes with the power maps P_k on G and K, the cohomology map f^* sends $H^n(G)_q$ to $H^n(K)_q$. Suppose for example that K is a closed connected subgroup of G and f is the inclusion map. Choose, as we may, a maximal torus T of G such that $S := T \cap K$ is a maximal torus of K. The restriction map $H(G) \to H(K)$ becomes, via (6.1), the map $[H(G/T) \otimes H(T)]^W \to [H(K/S) \otimes H(S)]^{W_K}$ induced by restriction on each factor, where W_K is the Weyl group of S in K.

(6.7) We close with the homology interpretation of (6.1), which says the homology map ψ_* induced by ψ is surjective. It follows that the homology of G is spanned by the cycles $[\psi(\bar{X}_w, T_I)] = \{gtg^{-1} : gT \in \bar{X}_w, t \in T_I\}$. Here $w \in W$, X_w is the Schubert cell (see (5.2)) and $T_I = \prod_{i \in I} T_i$, where $T = T_1 \times \cdots \times T_l$, with each $T_i \simeq S^1$. Using the results in [BGG], one can explicitly write down the action of W on $H_*(G/T)$ in terms of the Schubert cell basis, and this leads, in principle, to the linear relations in $H_*(G)$ satisfied by the cycles $[\psi(\bar{X}_w, T_I)]$.

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