7. COMMENTS

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In this section we give some details on the construction and on the proof of uniqueness of the even unimodular lattices of rank 32 with root systems $\mathbf{8A_1} \boxplus \mathbf{8A_3}$, $\mathbf{10A_2} \boxplus \mathbf{2E_6}$, $\mathbf{13A_2} \boxplus \mathbf{E_6}$, and $\mathbf{8A_4}$.

The first example, $\mathbf{8A_1} \boxplus \mathbf{8A_3}$, involves a rather heavy analysis, requiring some overview of the self-orthogonal codes in $T(\mathbf{8A_3})$ which is also necessary in order to treat the other root systems containing $\mathbf{8A_3}$.

The last three examples are hopefully more attractive.

(1) $\mathbf{8A_1} \boxplus \mathbf{8A_3}$

Here we have deficiency 8 and any metabolizer M must be of order 2^{12}.

If M is an admissible metabolizer and $P = P(x, y)$ its weight enumerator polynomial, the duality theorem of Section 4 provides an underdetermined linear system for the coefficients of P. The coefficients c, α, β, γ of x^6y^8, x^8y^6, x^8y^7 and x^8y^8 respectively can be taken as parameters and all other coefficients are then linear expressions in c, α, β, γ.

Let the polynomial P be

$$P(x, y) = 1 + c_1y^4 + c_2y^5 + c_3y^6 + c_4y^7 + c_5y^8 + \dots ,$$

where the dots stand for the terms which are divisible by x.

Then, the coefficients $c_1, \dots, c_5$ satisfy the equations

$$\begin{aligned} c_1 &= -37 + \alpha + 2\beta + 3\gamma, \\ c_2 &= 68 - 2\alpha - 3\beta - 4\gamma, \\ c_3 &= \alpha, \\ c_4 &= \beta, \\ c_5 &= \gamma. \end{aligned}$$

This shows that $1 + c_1 + c_2 + c_3 + c_4 + c_5 = 32$. If $M \subset T(\mathbf{8A_1} \boxplus \mathbf{8A_3})$ is an admissible metabolizer, then $1 + c_1y^4 + c_2y^5 + c_3y^6 + c_4y^7 + c_5y^8$ can be interpreted as the weight enumerator of $N = M \cap T(\mathbf{8A_3})$. Thus $|N| = 32$.

STEP 1. We will first show that N is uniquely determined up to a (norm preserving) automorphism of $T(\mathbf{8A_3})$.

Let $N' = N \cap 2T(\mathbf{8A_3})$. Consider the exact sequence

$$0 \to N' \to N \xrightarrow{\pi} N'' \to 0 ,$$

where π is the restriction to N of the projection $T(\mathbf{8A}_3) \to T(\mathbf{8A}_3)/2T(\mathbf{8A}_3)$, and $N'' = \pi(N) \subset T(\mathbf{8A}_3)/2T(\mathbf{8A}_3)$.

The map $\psi : N'' \to N'$ given by $\psi(x) = 2y$, where $\pi(y) = x$ is well defined, linear and injective. Hence, $|N''| \leqslant |N'|$ and since $|N| = |N'| \cdot |N''|$, it follows that there are 2 cases to be examined:

$$|N'| = 16 \quad \text{and} \quad |N''| = 2\,, \tag{1}$$

$$|N'| = 8 \quad \text{and} \quad |N''| = 4\,, \tag{2}$$

In case (1), there is just one possibility for N', namely

$$\begin{aligned} N' = < &(2,2,2,2,0,0,0,0),\ (2,2,0,0,2,2,0,0),\\ &(2,2,0,0,0,0,2,2),\ (2,0,2,0,2,0,2,0) > \end{aligned}$$

and there are 2 corresponding possibilities for N, depending on whether $\psi(N'') = <(2,2,2,2,2,2,2,2)>$ or $\psi(N'') = <(2,2,2,2,0,0,0,0)>$. Note that there is a single orbit of vectors of weight 4 under the group of permutations of the 8 coordinates in $T(\mathbf{8A}_3)$ preserving N'.

The 2 cases are specified by $N = N_1$ or N_2, where

$$\begin{aligned} N_1 = < &(1,1,1,1,1,1,1,1),\ (2,2,2,2,0,0,0,0),\\ &(2,2,0,0,2,2,0,0),\ (2,0,2,0,2,0,2,0) >, \end{aligned}$$

and

$$\begin{aligned} N_2 = < &(1,1,1,1,2,0,0,0),\ (2,2,0,0,2,2,0,0),\\ &(2,2,0,0,0,0,2,2),\ (2,0,2,0,2,0,2,0) >. \end{aligned}$$

For N_1, the weight polynomial is

$$P_1(0,y) = 1 + 14y^4 + 17y^8\,.$$

For N_2, the weight polynomial is

$$P_2(0,y) = 1 + 14y^4 + 8y^5 + 8y^7 + y^8\,.$$

However, in the second case, the polynomial coefficients of $P_2(0,y)$ would imply

$$\alpha = 0, \qquad \beta = 8, \qquad \gamma = 1$$

and thus $c_1 = -18$ for the coefficient of y^4 in $P(x,y)$. This case is therefore impossible and we retain only the possibility $N = N_1$ and

$$P_N(0,y) = 1 + 14y^4 + 17y^8\,.$$

As we shall see, it will actually turn out that the above subgroup N_1 is the only acceptable choice for $N = M \cap T(\mathbf{8A}_3)$.

In case (2), i.e. $|N'| = 8$, $|N''| = 4$, the possibilities for the weight polynomial of N' are

$$(2.1) \qquad P_{N'} = 1 + 5y^4 + 2y^6, \text{ or}$$

$$(2.2) \qquad P_{N'} = 1 + 6y^4 + y^8, \text{ or}$$

$$(2.3) \qquad P_{N'} = 1 + 7y^4 .$$

Moreover, in each case, N' is unique up to permutation of coordinates:

(2.1) $N' = <(2,2,2,2,2,2,0,0), (0,0,2,2,2,2,2,2), (2,0,2,0,2,0,2,0)>$,

(2.2) $N' = <(2,2,2,2,0,0,0,0), (0,0,0,0,2,2,2,2), (2,2,0,0,2,2,0,0)>$,

(2.3) $N' = <(2,2,2,2,0,0,0,0), (2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)>$.

In these cases, the image of $\psi : N'' \to N'$ is a plane i.e. $|\psi(N'')| = 4$ and since the admissible vectors of weight 6 in $T(\mathbf{8A}_3)$ are not divisible by 2 in the set of admissible vectors, it follows that $\psi(N'')$ contains only vectors of weight 0, 4 or 8.

In case (2.1), there is just one orbit of planes with all non-zero vectors of weight 4 under the action of the group of permutation of coordinates preserving N', namely the orbit of $<(2,2,0,0,0,0,2,2), (2,2,0,2,0,2,0,2,0)>$. However, it is easy to see that none of the admissible vectors $\upsilon \in T(\mathbf{8A}_3)$ such that $2\upsilon = (2,0,2,0,2,0,2,0)$, is compatible with N'. Typically, if $\upsilon = (1,2,1,0,1,0,1,0)$, then $\upsilon + (2,2,2,2,2,2,0,0) = (3,0,3,2,3,2,1,0)$ which has norm 5 and therefore is not admissible. Thus, in fact, case (2.1) cannot occur.

In case (2.2), where

$$N' = <(2,2,2,2,2,2,2,2), (2,2,2,2,0,0,0,0), (2,2,0,0,2,2,0,0)>$$

there are 2 orbits of planes in N' under the action of the automorphism group of N':

— The orbit $[u_1, u_2]$, $[u_1, u_3]$, $[u_1, u_2 + u_3]$ consisting of the planes containing $u_1 = (2,2,2,2,2,2,2,2)$ which is fixed by every automorphism.

— The orbit consisting of the planes $[u_2, u_3]$, $[u_1 + u_2, u_3]$, $[u_2, u_1 + u_3]$, $[u_1 + u_2, u_1 + u_3]$ not containing u_1.

Here, we have set $u_2 = (2,2,2,2,0,0,0,0)$ and $u_3 = (2,2,0,0,2,2,0,0)$.

Thus, we have two possible choices for the plane $\psi(N'')$, namely $[u_1, u_2]$ or $[u_2, u_3]$.

If $\psi(N'') = [u_1, u_2]$ is chosen, an enumeration of the possibilities shows that we can then assume N to be of the form

$$N = <(1,1,1,1,1,1,1,1), (1,1,1,3,0,2,0,0), (2,2,0,0,2,2,0,0)>.$$

The resulting weight polynomial for N, namely

$$P_N = 1 + 6y^4 + 8y^5 + 8y^7 + 9y^8$$

determines the coefficients α, β, γ as

$$\alpha = 0, \quad \beta = 8, \quad \gamma = 9 ,$$

and then, throwing in the monomials containing x, P_M becomes

$$\begin{aligned} P_M(x, y) = 1 &+ 6y^4 + 8y^5 + 8y^7 + 9y^8 + 24x^2y^3 + cx^2y^4 \\ &+ (400 - 4c)x^2y^5 + 6cx^2y^6 + (472 - 4c)x^2y^7 + cx^2y^8 \\ &+ 32x^4y^2 + (344 - 2c)x^4y^4 + (112 + 8c)x^4y^5 \\ &+ (1232 - 12c)x^4y^6 + (112 + 8c)x^4y^7 + (408 + 2c)x^4y^8 \\ &+ 24x^6y^3 + cx^8y^4 + 8x^8y^5 + 8x^8y^7 + 9x^8y^8, \end{aligned}$$

where c still has to be determined.

In order to calculate c, we examine the possible vectors of weight x^2y^7 in M. It is easy to see, considering the norm, that the only candidates must have the form $(1, 1, 0, 0, 0, 0, 0, 0; 2, 2, 2, 2, 2, 2, 2, 0)$ up to permutation of coordinates. But it is immediate that any such vector fails to be compatible with the vector $(0, 0, 0, 0, 0, 0, 0, 0; 2, 2, 2, 2, 2, 2, 2, 2) \in N \subset M$ because their sum would have norm 2. Therefore, the coefficient of x^2y^7 in P_M must be 0.

This forces $c = 118$. Unfortunately, the coefficient of x^2y^5 then becomes negative. Hence, there is no admissible metabolizer with this choice of $N = M \cap T(8\mathbf{A}_3)$.

The other choice (still under case (2.2)) is $\psi(N'') = [u_2, u_3]$. Here, an examination of the possible choices for N leads to either

$$N = \langle (1, 1, 1, 1, 2, 0, 0, 0), (1, 1, 2, 2, 1, 1, 0, 2), (0, 0, 0, 0, 2, 2, 2, 2) \rangle ,$$

or

$$N = \langle (1, 1, 1, 1, 2, 0, 0, 0), (1, 3, 0, 2, 1, 1, 0, 0), (0, 0, 0, 0, 2, 2, 2, 2) \rangle .$$

In both cases, the weight polynomial for N is

$$P_N = 1 + 6y^4 + 12y^5 + 12y^7 + y^8 ,$$

and this determines the parameters $\alpha = 0$, $\beta = 12$, $\gamma = 1$, contradicting the equation $c_1 = -37 + \alpha + 2\beta + 3\gamma$.

There remains the case (2.3), where

$$N' = \langle (2, 2, 2, 2, 0, 0, 0, 0), (2, 2, 0, 0, 2, 2, 0, 0), (2, 0, 2, 0, 2, 0, 2, 0) \rangle .$$

In this case, it is easy to see that there is just one orbit of planes in N' under the action of the group of coordinate permutations preserving N'. Hence, we may assume $\psi(N'') = [u_1, u_2]$, where $u_1 = (2,2,2,2,0,0,0,0)$ and $u_2 = (2,2,0,0,2,2,0,0)$ and there are 4 choices for N:

They are $< N_i, u_3 >$, $i = 1,2,3,4$, where $u_3 = (2,0,2,0,2,0,2,0)$ and

$$\begin{aligned} N_1 &= <(1,1,1,1,2,0,0,0), (1,1,0,0,1,1,2,0)>, \\ N_2 &= <(1,1,1,1,2,0,0,0), (1,1,0,0,1,1,0,2)>, \\ N_3 &= <(1,1,1,1,0,0,0,2), (1,1,2,0,1,1,0,0)>, \\ N_4 &= <(1,1,1,1,0,0,0,2), (1,1,2,0,1,1,2,2)>. \end{aligned}$$

The resulting polynomials P_N are $1 + 7y^4 + 18y^5 + 6y^7$ in the first case, and $1 + 7y^4 + 10y^5 + 14y^7$ in the last 3 cases.

In both instances, the values of the parameters α, β, γ contradict the equation for c_1.

Summarizing this first phase of the analysis, we necessarily have

$$\begin{aligned} N = \ <&(1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0), \\ &(2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)>, \end{aligned}$$

and the vanishing of the coefficient of x^2y^7 (because any vector of weight x^2y^7 is incompatible with $(0,0,0,0,0,0,0,0;2,2,2,2,2,2,2,2) \in N$) forces the weight polynomial to be as announced:

$$\begin{aligned} P(x,y) = 1 &+ x^8 + 56x^4y^2 + 14y^4 + 112x^2y^4 + 112x^4y^4 + 112x^6y^4 \\ &+ 14x^8y^4 + 896x^4y^5 + 672x^2y^6 + 56x^4y^6 + 672x^6y^6 \\ &+ 896x^4y^7 + 17y^8 + 112x^2y^8 + 224x^4y^8 + 112x^6y^8 + 17x^8y^8. \end{aligned}$$

Thus the weight enumerator of any putative admissible metabolizer is uniquely determined after all, and more importantly $N = M \cap T(\mathbf{8A}_3)$ is uniquely determined as

$$\begin{aligned} N = \ <&(1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0), \\ &(2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)>. \end{aligned}$$

STEP 2. Now, since $|M| = 2^{12}$ and $|N| = 2^5$ the projection of any metabolizer M into $T(\mathbf{8A}_1)$ must be a 7-dimensional subspace. Since the polynomial P_M contains only monomials with x to an even power, the projection of M into $T(\mathbf{8A}_1)$ consists exactly of the vectors of even weight. Let $e_i \in T(\mathbf{8A}_1) = \mathbf{F}_2^8$ be the vectors with coordinates i and $i+1$ equal to 1 and all others $0 (i = 1, \ldots, 7)$. If $v \in T(\mathbf{8A}_3)$, we use the (hopefully) self-explanatory notation $e_i + v \in T(\mathbf{8A}_1) \boxplus T(\mathbf{8A}_3)$. Obviously, M admits a system of generators consisting of vectors of the form $e_k + \upsilon_{i_k}$, $k = 1, \ldots, 7$ together with N.

There is a list of 28 classes $\upsilon + N$ modulo N of vectors υ such that $e_i + \upsilon$ is compatible with N, i.e. such that the subgroup of $T(\mathbf{8A}_1) \boxplus T(\mathbf{8A}_3)$ generated by $e_i + \upsilon$ and N consists entirely of admissible vectors.

Each class has a representative with all non-zero coordinates equal to 1 or 3 and first non-zero coordinate equal to 1. The list reads as follows:

$$\begin{aligned}
\upsilon_0 &= (0,0,0,0,1,1,1,1), & \upsilon_7 &= (0,0,0,0,1,1,3,3),\\
\upsilon_1 &= (0,0,1,1,1,1,0,0), & \upsilon_8 &= (0,0,1,1,3,3,0,0),\\
\upsilon_2 &= (0,0,1,1,0,0,1,1), & \upsilon_9 &= (0,0,1,1,0,0,3,3),\\
\upsilon_3 &= (0,1,0,1,0,1,0,1), & \upsilon_{10} &= (0,1,0,1,0,3,0,3),\\
\upsilon_4 &= (0,1,0,1,1,0,1,0), & \upsilon_{11} &= (0,1,0,1,3,0,0,3),\\
\upsilon_5 &= (0,1,1,0,1,0,0,1), & \upsilon_{12} &= (0,1,1,0,3,0,0,3),\\
\upsilon_6 &= (0,1,1,0,0,1,1,0), & \upsilon_{13} &= (0,1,1,0,0,3,3,0),
\end{aligned}$$

$$\begin{aligned}
\upsilon_{14} &= (0,0,0,0,1,3,1,3), & \upsilon_{21} &= (0,0,0,0,1,3,3,1),\\
\upsilon_{15} &= (0,0,1,3,1,3,0,0), & \upsilon_{22} &= (0,0,1,3,3,1,0,0),\\
\upsilon_{16} &= (0,0,1,3,0,0,1,3), & \upsilon_{23} &= (0,0,1,3,0,0,3,1),\\
\upsilon_{17} &= (0,1,0,3,0,1,0,3), & \upsilon_{24} &= (0,1,0,3,0,3,0,1),\\
\upsilon_{18} &= (0,1,0,3,1,0,3,0), & \upsilon_{25} &= (0,1,0,3,3,0,1,0),\\
\upsilon_{19} &= (0,1,3,0,1,0,0,3), & \upsilon_{26} &= (0,1,3,0,3,0,0,1),\\
\upsilon_{20} &= (0,1,3,0,0,1,3,0), & \upsilon_{27} &= (0,1,3,0,0,3,1,0).
\end{aligned}$$

Thus any admissible metabolizer M is generated by $N \subset T(\mathbf{8A}_3) \subset T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$, where

$$\begin{aligned} N = \langle &(1,1,1,1,1,1,1,1), (2,2,2,2,0,0,0,0),\\ &(2,2,0,0,2,2,0,0), (2,0,2,0,2,0,2,0)\rangle, \end{aligned}$$

and 7 vectors of the form

$$s_1 = e_1 + \upsilon_{k_1}, s_2 = e_2 + \upsilon_{k_2}, \ldots, s_7 = e_7 + \upsilon_{k_7},$$

where $\upsilon_{k_1}, \upsilon_{k_2}, \ldots, \upsilon_{k_7}$ are taken from the above list.

A septet $(k_1, \ldots, k_7)$ such that the subgroup $M = \langle s_1, \ldots, s_7\rangle + N$ is an admissible metabolizer (i.e. consisting only of vectors of integral, even norm $\neq 2$) will be called an *admissible septet* and the corresponding metabolizer $\langle s_1, \ldots, s_7\rangle + N$ will be denoted $M(i_1, \ldots, i_7)$.

In order to determine the admissible septets it is not necessary to handle the $\binom{28}{7} \times 7! = 5967561600$ cases. One first makes a list P_0 of pairs (i, j)

such that

$$M_{i,j} = \langle e_1 + v_i, e_3 + v_j \rangle + N$$

is an admissible subgroup. The list P_0 contains 210 unordered pairs (420 if (i, j) and (j, i) are counted for 2).

The machine can then easily sort out the (unordered) quadruples (i, j, k, l) such that the 6 pairs $(i, j), (i, k), \ldots, (k, l)$ belong to P_0, a condition which is necessary for (i, j, k, l) to appear as $i = i_1, j = i_3, k = i_5, l = i_7$ in some admissible septet $(i_1, i_2, i_3, \ldots, i_7)$. A list Q of 105 quadruples comes out.

Note that if $(i_1, i_2, \ldots, i_7)$ is an admissible septet and (i_1', i_3', i_5', i_7') is any permutation of (i_1, i_3, i_5, i_7), there is a new triple (i_2', i_4', i_6') such that $(i_1', i_2', i_3', \ldots, i_6', i_7')$ is again an admissible septet and the corresponding metabolizers M, M' yield isomorphic lattices.

For instance, if $M = \langle e_1 + v_{i_1}, \ldots, e_7 + v_{i_7} \rangle + N$, then the permutation $\pi = (1\ 3)\ (2\ 4)$ on the first 8 coordinates (permuting the factors $T(\mathbf{A}_1)$) and leaving $T(\mathbf{8A}_3)$ fixed, carries M to

$$\begin{aligned} M' &= \langle e_3 + v_{i_1}, e_1 + e_2 + e_3 + v_{i_2}, e_1 + v_{i_3}, e_4 + v_{i_4}, \ldots, e_7 + v_{i_7} \rangle + N \\ &= \langle e_1 + v_{i_3}, e_2 + v_{i_2}', e_3 + v_{i_1}, e_4 + v_{i_4}, \ldots, e_7 + v_{i_7} \rangle + N, \end{aligned}$$

where $v_{i_2}' = v_{i_1} + v_{i_2} + v_{i_3}$. Then, v_{i_2}' must be a vector $v_{i_2'}$ of the above basic list (up to addition of a vector of N). Therefore, $(i_3, i_2', i_1, i_4, i_5, i_6, i_7)$ is an admissible septet. Thus, any equivalence class of admissible metabolizer can be represented by a septet $(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ such that $i_1 < i_3 < i_5 < i_7$.

Now, let G be the group of permutations of the last 8 coordinates in $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$ generated by

$$\alpha = (1\ 2)\ (3\ 4), \quad \beta = (3\ 5)\ (4\ 6), \quad \gamma = (1\ 7)\ (2\ 8), \quad \rho = (1\ 6)\ (3\ 8)$$

permuting the 8 factors $T(\mathbf{A}_3)$ in $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$.

The group G has order 1344 and it operates on the set of classes *mod* N of the 28 vectors of the above basic list. It operates therefore also on the set Q of quadruples. The 105 quadruples forming Q are then divided into 3 orbits under this action, represented by the quadruples

$$\begin{aligned} q_0 &= (0, 7, 14, 21) \text{ with } Gq_0 \text{ of cardinality } 7, \\ q_1 &= (0, 7, 16, 23) \text{ with } Gq_1 \text{ of cardinality } 42, \\ q_2 &= (5, 10, 20, 25) \text{ with } Gq_2 \text{ of cardinality } 56. \end{aligned}$$

Next, let P_1 be the set of pairs (i, j) such that

$$M_{i,j}' = \langle e_1 + v_i, e_2 + v_j \rangle + N$$

is an admissible subgroup of $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$, i.e. consisting entirely of vectors v such that the norm $\mathbf{n}(v)$ of v is an even integer $\neq 2$. The set P_1 contains 336 ordered pairs (obviously $(i, j) \in P_1$ implies $(j, i) \in P_1$). Any admissible septet $(i_1, \ldots, i_7)$ must be such that $(i_1, i_3, i_5, i_7) \in Q$, and $(i_k, i_{k+1}) \in P_1$ for $k = 1, \ldots, 6$, in addition to $(i_k, i_l) \in P_0$ for $|k - l| \geqslant 2$.

Given a quadruple $q = (i_1, i_3, i_5, i_7) \in Q$, it is not hard to sort out the set T_q of triples (i_2, i_4, i_6) such that $(i_1, i_2, \ldots, i_7)$ satisfies all the conditions on the pairs (i_k, i_l). We need to do this in fact only for the above 3 quadruples q_0, q_1, q_2, since any admissible septet can be carried by the action of G to a septet $(i_1, i_2, \ldots, i_7)$ completing q_0, q_1 or q_2 in the sense that $(i_1, i_3, i_5, i_7) = q_0, q_1$ or q_2.

It turns out that for each of these 3 quadruples $q = (i_1, i_3, i_5, i_7)$, there are 16 triples in the set T_q.

The resulting set of 48 septets can in fact still be reduced using the action of G. The subgroups of G fixing q_0, q_1 or q_2 are respectively of order 8, 4 and 1 and we are left with the following septets:

$$(0, 1, 7, 20, 14, 22, 21), \qquad (0, 1, 7, 20, 14, 23, 21)$$

completing q_0;

$$(0, 1, 7, 20, 16, 21, 23), \qquad (0, 1, 7, 20, 16, 22, 23)$$
$$(0, 9, 7, 20, 16, 21, 23), \qquad (0, 9, 7, 20, 16, 22, 23)$$

completing q_1;

and with the quadruple $q_2 = (5, 10, 20, 25)$ there are the 16 triples

(0, 14, 7),	(0, 14, 17),	(0, 19, 16),	(0, 19, 26),
(13, 14, 7),	(13, 14, 17),	(13, 19, 16),	(13, 19, 26),
(4, 11, 7),	(4, 11, 17),	(4, 8, 16),	(4, 8, 26),
(23, 11, 7),	(23, 11, 17),	(23, 8, 16),	(23, 8, 26),

forming the septets (5, 0, 10, 14, 20, 7, 25), etc.

Denote by $M(i_1, i_2, i_3, i_4, i_5, i_6, i_7)$ the subgroup

$$M(i_1, \ldots, i_7) = \langle e_1 + v_{i_1}, \ldots, e_7 + v_{i_7} \rangle + N.$$

We finish exploiting the operations of the permutation group S_8 acting on $T(\mathbf{8A}_1 \boxplus \mathbf{8A}_3)$ by permuting the first 8 coordinates.

It is easy to check that $\sigma_1 = (1\ 2) \in S_8$ acts on admissible metabolizers of the form $M(i_1, i_2, i_3, \ldots, i_7)$ by

$$\sigma_1 M(i_1, i_2, i_3, \ldots, i_7) = M(i_1, i_2', i_3, \ldots, i_7)\ ,$$

where $v_{i'_2}$ is the uniquely determined element in the basic list such that $v_{i'_2} \equiv v_{i_1} + v_{i_2}$ modulo N.

Similarly,

$$\sigma_k \, M(i_1, \dots, i_7) = M(i'_1, \dots, i'_7) \, ,$$

where $i'_l = i_l$ for $l \neq k - 1, \, k + 1$ and

$$v_{i'_{k-1}} \equiv v_{i_{k-1}} + v_{i_k} \text{ modulo } N, v_{i'_{k+1}} \equiv v_{i_k} + v_{i_{k+1}} \text{ modulo } N \, ,$$

for $k = 1, 2, \dots, 6$;

$$\sigma_7 M(i_1, \dots, i_7) = M(i_1, \dots, i_5, i'_6, i_7) \, ,$$

where $v_{i'_6} \equiv v_{i_6} + v_{i_7}$ modulo N.

Using $\sigma_1, \sigma_3, \sigma_5$ and σ_7 one first observes that all $M(i_1, i_2, \dots, i_7)$ with the same quadruple $q = (i_1, i_3, i_5, i_7)$ are equivalent. Hence, the equivalence class of any admissible metabolizer is detected by its basic quadruple which can be q_0, q_1 or q_2. However, the permutation σ_6 carries $M(0, 1, 7, 20, 14, 22, 21)$ to $M(0, 1, 7, 20, 16, 22, 21)$. Similarly, the permutation $\pi = (7\,4\,5\,6\,3\,2\,1\,8)$ takes $M(5, 0, 10, 14, 20, 7, 25)$ to $M(0, 8, 7, 27, 14, 16, 21)$ which is equivalent to $M(0, 1, 7, 20, 14, 22, 21)$.

It is easy to let the machine verify that $M(0, 1, 7, 20, 14, 22, 21)$ actually is an admissible metabolizer and to pass from it to the filling set displayed in the table.

Thus, there is a single isomorphism class of 32-dimensional even, unimodular lattice with root system $\mathbf{8A_1} \boxplus \mathbf{8A_3}$.

(2) $\mathbf{10A_2} \boxplus \mathbf{2E_6}$

The only weight enumerator polynomial $P(x, y)$ for an admissible metabolizer in $T(\mathbf{10A_2} \boxplus \mathbf{2E_6})$ which is compatible with the duality theorem is

$$P(x, y) = 1 + 60x^6 + 20x^9 + 60x^4y + 240x^7y + 24x^{10}y + 144x^5y^2 + 180x^8y^2 \, .$$

Thus in $T(\mathbf{10A_2}) = \mathbf{F}_3^{10}$, the intersection $M_0 = M \cap T(\mathbf{10A_2})$ contains exactly 10 pairs $\{x, -x\}$ of vectors of Hamming weight 9.

Two distinct such pairs $\{x, -x\}$ and $\{x', -x'\}$ cannot have their vanishing coordinate at the same place. Indeed, suppose that for some i, we have $x'_i = x_i = 0$. Set $J = \{j \in \{1, \dots, 10\} \mid x'_j = x_j \neq 0\}$ and $K = \{k \in \{1, \dots, 10\} \mid x'_k = -x_k \neq 0\}$. Then $|J| + |K| = 9$, and $w(x + x') = |J|$, $w(x - x') = |K|$. The polynomial says that $|J| \neq 3$, $|K| \neq 3$. Hence the only possibility is $\{|J|, |K|\} = \{0, 9\}$ and $x' = \pm x$.

By numbering the 10 pairs $\{x^{(1)}, -x^{(1)}\}, \ldots, \{x^{(10)}, -x^{(10)}\}$ correctly, we can thus assume that the i-th coordinate of $x^{(i)}$ is 0. Let us choose $\{0, -1, 1\}$ as integer representatives of the residue classes mod 3. The vectors $x^{(1)}, \ldots, x^{(10)}$ can be thought of as the (reduction mod 3 of the) rows of a 10×10 integral matrix C such that

$$c_{i,i} = 0, \qquad c_{i,j} = \pm 1 \text{ for } i \neq j .$$

I claim that C is a *conference matrix*, i.e. $C^t . C = C . C^t = 9I$, where I is the 10×10 unit matrix.

For $i \neq j$, let $S = \{s \in \{1, \ldots, 10\} \mid x_s^{(i)} = x_s^{(j)}\}$. Clearly $i, j \notin S$. Since $w(x^{(i)} + x^{(j)}) = 2 + |S|$, and $w(x^{(i)} - x^{(j)}) = 2 + (8 - |S|)$, and the only possible values are 6 or 9, we conclude that $|S| = 4$. It follows that the scalar product of two distinct rows of C is zero.

Up to conjugation by a signed permutation matrix there is exactly one 10×10 conference matrix. Thus M_0 is uniquely determined.

It is easy to verify that there is then no choice left for the last two filling vectors (up to isomorphism of the lattices).

(3) $\mathbf{13A_2 \boxplus E_6}$

Here, not only is the weight polynomial determined by the duality theorem, but if we single out one of the factors $T(\mathbf{A}_2)$, the polynomial $P(x_1, x_2, y)$ corresponding to the decomposition $\mathbf{12A_2 \boxplus A_2 \boxplus E_6}$ is still uniquely determined and reads

$$P(x_1, x_2, y) = 1 + 84x_1^6 + 152x_1^9 + 6x_1^{12} \\ + (\text{sum of monomials divisible by } x_2 \text{ or } y) .$$

This means that if M is an admissible metabolizer, then for any choice of coordinate (among the first 13) there must be exactly 3 pairs of vectors of weight 12 having precisely this coordinate zero.

It is then straightforward to see that we may assume these 3 pairs of vectors to be $\pm s_1, \pm s_2, \pm s_3$, where

$$\begin{aligned} s_1 &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0; 0), \\ s_2 &= (1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 0; 0), \\ s_3 &= (1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 2, 2, 0; 0). \end{aligned}$$

It now turns out that the vectors with vanishing 12-th coordinate in M can then be assumed to be

$$\begin{aligned} s_4 &= (1, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 0, 1; 0) \\ s_5 &= (1, 1, 2, 2, 2, 1, 2, 2, 1, 2, 2, 0, 1; 0) \\ s_1 - s_2 - s_3 + s_4 + s_5 &= (1, 2, 2, 2, 1, 1, 2, 1, 1, 1, 1, 0, 2; 0) \end{aligned}$$

and their opposites, where s_1, s_2, s_3, s_4, s_5 are linearly independent and form a basis of an admissible 5-dimensional subspace in $T(13\mathbf{A}_2)$.

Indeed, among the first 11 coordinates of these 6 vectors, there must be either 4 ones and 7 twos or 4 twos and 7 ones. Since we can change the sign of the last (13-th coordinate) at will, we may assume that s_4 has the form $(1^4, 2^7, 0, 1)$, meaning 4 ones and 7 twos among the first 11 coordinates.

From the list of $\binom{11}{4} = 330$ such vectors, a sublist of 27 vectors only are compatible with s_1, s_2, s_3. Moreover, these represent a single class modulo permutations of the coordinate indices $\{1, 2, 3\}$, $\{4, 5, 6\}$, $\{7, 8, 9\}$ which preserve the subspace generated by s_1, s_2, s_3. Having chosen

$$s_4 = (1, 2, 1, 2, 1, 2, 2, 1, 2, 2, 2, 0, 1; 0),$$

we must select among the remaining 26 vectors compatible with s_1, s_2, s_3 together with the 27 vectors of the form $(1^4, 2^7, 0, 2; 0)$, those which are compatible with s_1, s_2, s_3, s_4. Of these, only 8 candidate vectors come out. They form a single class modulo the group generated by the permutations (1 3), (4 6), (7 9). Hence, the choice of

$$s_5 = (1, 1, 2, 2, 2, 1, 2, 2, 1, 2, 2, 0, 1; 0)$$

is also essentially unique.

Observe that $M \cap T(13\mathbf{A}_2)$ has to be 6-dimensional because the sum of the coefficients of the monomials not containing y in the weight polynomial of M is $729 = 3^6$. The search for a 6-th and last basis vector for $M \cap T(13\mathbf{A}_2)$ shows that the choice is limited to

$$s_6 = (1, 1, 2, 1, 2, 2, 2, 1, 2, 2, 0, 2, 1; 0)$$

and its 6 transforms under the group of permutations of coordinates generated by the permutations (2 3) (5 6) (8 9) and (1 2 3) (4 5 6) (7 8 9) which preserves the subspace generated by s_1, s_2, s_3, s_4, s_5.

Thus, there is essentially only one choice for $M \cap T(13\mathbf{A}_2)$. The metabolizer M itself is then easily seen to be uniquely determined.

The transformation

$$\rho(x_0, \dots, x_{12}) = (-x_2, -x_{11}, x_7, -x_0, x_8, -x_1, x_5, x_4, -x_9, -x_{10}, x_3, x_6, x_{12})$$

carries M_0 as just described to the cyclic code of the table in Section 6.

(4) **$8\mathbf{A}_4$**

Let $e_1 = (1,0,0,0)$, $e_2 = (0,1,0,0)$, $e_3 = (0,0,1,0)$, $e_4 = (0,0,0,1)$. Any metabolizer must have a basis of the form $\{e_i + v_i, i = 1,2,3,4\}$ for some vectors $v_i \in \mathbf{F}_5^4$ of weight 3 or 4.

Hence, we may assume that the first basis vector is either $s_1 = e_1 + (1,1,1,1)$ or $t_1 = e_1 + (0,1,2,2)$.

If we start with s_1, there are essentially only 2 ways of completing s_1 to an admissible metabolizer with 3 vectors forming with s_1 the rows of the matrix S exhibited in the table and the matrix S':

$$S' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}.$$

If we start with t_1 there is essentially only one way to complete to a metabolizer:

$$S'' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 0 & 3 & 3 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 & 0 \end{pmatrix}.$$

All these metabolizers are equivalent. The permutation ρ defined by

$$\rho(x_0, \ldots, x_7) = (x_4, x_1, x_2, -x_3, x_7, x_5, x_6, x_0)$$

sends S' to S and σ defined by

$$\sigma(x_0, \ldots, x_7) = (x_5, x_1, x_4, x_0, x_7, x_2, x_3, x_6)$$

sends S'' to S.

Thus the lattice described by the filling set S is the only one with the root system $8\mathbf{A}_4$.

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