Section 3: Convex Hulls

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Next, we compare $t_* s_* (t^{-1})_* (s^{-1})_*$ to the identity map on e_k. Since $s_* (t^{-1})_*$ and $(t^{-1})_* s_*$ are 72δ close to the same translation, and translations commute, we have that $t_* s_* (t^{-1})_* (s^{-1})_*$ and $t_* (t^{-1})_* s_* (s^{-1})_*$ are $(144 + 24)\delta$ close. Moreover, $t_* (t^{-1})_*$ and $s_* (s^{-1})_*$ are 36δ close to the identity. Thus $t_* (t^{-1})_* s_* (s^{-1})_*$ is 108δ close to the identity. Hence $t_* s_* (t^{-1})_* (s^{-1})_*$ is 276δ close to the identity. Combining this with the estimate in the previous paragraph we have that the restriction of $(tst^{-1}s^{-1})$ to e_k is 532δ close to the identity on e_k. Therefore, a vertex close to the midpoint of e_k is moved by less than $532\delta + 2$ by $tst^{-1}s^{-1}$. Thus $tst^{-1}s^{-1}$ lies in the ball of radius $532\delta + 2$ about the identity in Γ, and we have the desired bound on the number of commutators in the arbitrary finite subset $P \subset G$.

Now Lemma 2.7 implies that G is virtually abelian. But every abelian subgroup of a hyperbolic group is virtually cyclic. Hence the segment stabilizers for the action of Γ on X_∞ must be virtually cyclic. This completes the proof of Paulin's theorem. $\square$

SECTION 3: CONVEX HULLS

A subset Σ of a geodesic metric space X is said to be *geodesically convex* if for all $p, q \in \Sigma$ every geodesic segment from p to q is completely contained in Σ. Given a bounded set $Y \subset X$, perhaps the most natural way to define its convex hull is as the intersection of all geodesically convex sets containing Y.

If X is simply connected and non-positively curved then round balls are geodesically convex and hence the convex hull of a bounded set is bounded. However, for more general geodesic metric spaces, even δ-hyperbolic spaces, it may happen that the convex hull of a finite set is the whole of the ambient space X. The following example illustrates how general this problem is.

3.1 PROPOSITION. *Given any finitely generated group Γ there exists a finite generating set S and a finite subset $Y \subset \Gamma$ such that the convex hull of Y in the Cayley graph $X(\Gamma, S)$ is the whole of $X(\Gamma, S)$.*

Proof. Let A be any finite generating set for Γ, and take S to be the set of those elements of Γ which are a distance 1 or 4 from the identity in the Cayley graph of Γ with respect to S. Let Y be the set of elements of Γ which are a distance at most 3 away from the identity in the Cayley graph associated to A.

Notice that the convex hull of Y with respect to S contains the ball of radius 4 as measured in the A metric. Furthermore, a simple induction shows that if this convex hull contains the balls of radius n and $n + 3$ about the identity (as measured in the metric associated to A) then it contains the ball of radius $n + 4$. Thus the convex hull of Y is the whole of $X(\Gamma, S)$. $\square$

SECTION 4: CONCLUDING REMARKS

The type of limit spaces which we considered in Section 2 first arose in work of Morgan and Shalen in which they reinterpreted and generalized Thurston's compactification of Teichmuller space (see [Sha]). The particular topology with respect to which limits are taken in that setting is equivalent to what Paulin has termed "Equivariant Gromov convergence" (see [P1, 2]). It can be shown that the limit tree which we constructed in Section 2 is also a limit in the sense of this topology. We recall Paulin's recent definition:

4.1 DEFINITION. *A sequence of metric spaces* Y_n *which are equipped with actions by isometries of a fixed group* Γ, *converge to a metric space* Y, *which is also equipped with an action of* Γ *by isometries, if and only if, given any finite set* $K \subset Y$, *any* $\varepsilon > 0$, *and any finite subset* $P \subset \Gamma$, *for sufficiently large* n, *one can find subsets* $K_n \subset Y_n$ *and bijections* $x_n \mapsto x$ *from* K_n *to* K, *such that*

$$|d(\gamma x, y) - d_n(\gamma x_n, y_n)| < \varepsilon$$

for all $x, y \in K$ *and all* $\gamma \in P$.

Limits are not unique in this topology, even if one allows only limit spaces which are complete (cf. [P2], p. 55).

The technique of Equivariant Gromov convergence has been successfully applied in the following settings:

(1) $Y_n = \mathbf{H}^m$ for every integer n and the action of the (abstract) group Γ is discrete and varies with n;

(2) the spaces Y_n are $\mathbf{R}$-trees with isometric Γ-actions;

(3) each Y_n is equal to the Cayley graph of Γ with respect to a fixed set of generators and the action of Γ is left-multiplication twisted by a sequence of homomorphisms $\varphi_n : \Gamma \to \Gamma$.

The situation which we considered in Section 2 belongs to the third of the above cases. In the first two cases, the spaces under consideration enjoy strong convexity properties that allow one to form compact convex hulls of any finite