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THE OKA-PRINCIPLE FOR MAPPINGS BETWEEN RIEMANN SURFACES

by Jörg WINKELMANN

The Oka-principle is the philosophy that for Stein manifolds complex-analytic properties should parallel topological properties. In this spirit it is important to know under which conditions homotopy classes of continuous maps correspond to homotopy classes of holomorphic maps. The first basic result in this regard is due to Grauert [2] for maps from Stein spaces into complex Lie group bundles. For example, this is a key ingredient in Grauert's proof that the topological classification of complex vector bundles over Stein spaces is the same as the holomorphic one.

Over the years there have been numerous important applications and extensions of Grauert's Oka-principle. It is still of interest to determine if there is a significantly broader class of spaces where the principle remains valid. For example, Gromov has introduced "elliptic bundles" for this purpose [4].

Our goal here is to completely determine all pairs of Riemann surfaces for which this homotopy principle holds, i.e. for which every continuous map is homotopic to a holomorphic one.

THEOREM 1. *Let M, N be Riemann surfaces. Then every continuous map from M to N is homotopic to a holomorphic map in the following cases:*

1) *M or N is isomorphic to $\mathbf{C}$ or $\Delta = \{z \in \mathbf{C} : |z| < 1\}$ or $M \simeq \mathbf{P}_1 \neq N$.*

2) *M is non-compact and N is isomorphic to $\mathbf{P}_1$, $\mathbf{C}^*$ or a torus.*

3) *N is isomorphic to $\Delta^* = \Delta \setminus \{0\}$ and $M = \bar{M} \setminus \cup_i \bar{D}_i$ where $\bar{M}$ is a compact Riemann surface and D_i are closed disks (with radius bigger than zero).*

In all other cases there exists a continuous map from M to N which is not homotopic to a holomorphic map.

Next we give a description of the "other cases":

PROPOSITION 1. *The "other cases" of the theorem are the following:*

i) *M compact and $N \simeq \mathbf{P}_1$.*

ii) *M is compact and both N and M are not simply-connected.*

iii) *M is neither compact nor simply-connected and N is not isomorphic to a torus or one of the following surfaces: $\mathbf{P}_1, \mathbf{C}, \mathbf{C}^*, \Delta, \Delta^*$.*

iv) *$M = M_1 \backslash \{p\}$ and $N \simeq \Delta^*$.*

v) *$H_1(M)$ is not finitely generated and $N \simeq \Delta^*$.*

First we show that these two lists exhaust all possible pairs of Riemann surfaces. This is in fact an easy consequence of the well-known result below.

THEOREM 2. *Let M be a Riemann surface. Assume that $H_1(M)$ is finitely generated. Then there exists a compact Riemann surface $\bar{M}$, disjoint embedded closed disks $\bar{D}_i \subset \bar{M}$ and points p_j such that $M = \bar{M} \backslash \cup \bigcup_i \bar{D}_i \cup \bigcup_j \{p_j\}$.*

Proof (see also [7], 4.11.). Any Riemann surface M admits an exhaustion by compact bordered Riemann surfaces P_n with $P_n \subset\subset \overset{\circ}{P}_{n+1}$. This exhaustion may be choosen in such a way (see [1], I.29) that $\tau_n: H_1(P_n) \to H_1(P_{n+1})$ is always injective. Moreover either τ_n is an isomorphism or the rank increases. Thus $H_1(M)$ being finitely generated implies that $H_1(M) \simeq H_1(P_N)$ for some $N \geqslant 0$. Furthermore $M = P_{N-1} \cup \bigcup_i Q_i$ where the Q_i are disjoint open subsets with $H_1(Q_i) \simeq \mathbf{Z}$. From the uniformization theorem it follows that $\xi_i: Q_i \simeq A(r_i, 1)$ with $0 \leqslant r_i < 1$. (As usual $A(r, s) = \{z : r < |z| < s\}$). Let us consider $\xi_i(P_N \cap Q_i)$. Since P_N is compact, we may deduce $\xi_i(P_N \cap Q_i) \subset A(s_i, 1)$ with $r_i < s_i$. Using the embeddings $A(r_i, 1) \subset \Delta$ we now obtain the desired embedding of M in a compact Riemann surface $\bar{M}$. □

In the sequel $[M : N]$ always denotes the set of all (free) homotopy classes of continuous maps from M to N. We will need the classification of homotopy classes of continuous maps between Riemann surfaces.

THEOREM 3. *Let M, N be Riemann surfaces. If $N \simeq \mathbf{P}_1$, then $[M : N] \simeq H^2(M, \mathbf{Z})$. If M is compact, then $H^2(M, \mathbf{Z}) \simeq \mathbf{Z}$ and the continuous maps are classified up to homotopy by the Brouwer degree. For non-compact M we have $H^2(M, \mathbf{Z}) = \{0\}$.*

If $N \neq \mathbf{P}_1$, then $[M : N] \simeq Hom(\pi_1(M), \pi_1(N))$.

Proof. This is standard algebraic topology (see e.g. [6]). The first statement is a special case of the Hopf classification. The last assertion holds, because due to the uniformization theorem $N \not\simeq \mathbf{P}_1$ implies that N is an Eilenberg-MacLane space. □

We start with the proof of the positive statements. First note that in case 1) any continuous map is homotopic to a constant map.

Case 2) may be deduced from the general result of Grauert [2]. However, in this simple case there is an alternative approach. If $N \simeq \mathbf{P}_1$, then $[M:\mathbf{P}_1] \simeq H^2(M, \mathbf{Z})$. But $H^2(M, \mathbf{Z}) = 0$ for a non-compact Riemann surface. Thus every continuous map is homotopic to a constant map for $N \simeq \mathbf{P}_1$. For $N \simeq \mathbf{C}^*$ note that the exponential sequence.

$$0 \to \mathbf{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

yields a surjective map $\delta : \Gamma(M, \mathcal{O}^*) \to H^1(M, \mathbf{Z})$, because $H^1(M, \mathcal{O}) = \{0\}$ for a non-compact Riemann surface M. Now it follows from $Hol(M, \mathbf{C}^*) = \mathcal{O}^*(M)$ and

$$H^1(M, \mathbf{Z}) \simeq Hom(\pi_1(M), \mathbf{Z}) \simeq Hom(\pi_1(M), \pi_1(\mathbf{C}^*))$$

that every continuous map from M to $\mathbf{C}^*$ is homotopic to a holomorphic one.

Finally consider the subcase where N is a torus. Observe that for any one-dimensional torus T there is a projection $\tau : \mathbf{C}^* \times \mathbf{C}^* \to T$ which is holomorphic and a homotopy equivalence. Thus the statement for $\mathbf{C}^*$ implies the statement for tori.

Proof of case 3). Let D_i' be relatively compact subdisks of D_i and $M_1 = \bar{M} \setminus \cup_i \bar{D}_i'$. Now M is homotopic to M_1 and M is relatively compact in M_1. Any continuous map $f: M \to \Delta^*$ is homotopic to the restriction of a continuous map $F: M_1 \to \mathbf{C}^*$. Now $F \sim G$ for some holomorphic map $G: M_1 \to \mathbf{C}^*$. Note that $G(\bar{M})$ is compact. Hence $\lambda G(\bar{M}) \subset \Delta^*$ for some λ. □

This completes the proof of the positive results. Now we have to prove the negative statements.

We recall a standard fact about Riemann surfaces (see e.g. [1]):

LEMMA 1. *Let M be a Riemann surface. If $H_1(M) = \{0\}$ or $H_1(M) \simeq \mathbf{Z}$, then $H_1(M) \simeq \pi_1(M)$.*

COROLLARY. *Let M, N be Riemann surfaces. Assume that neither M nor N is simply-connected.*

Then $Hom(\pi_1(M), \pi_1(N)) \neq \{0\}$.

Proof. By the preceding lemma $\pi_1(M) \neq \{0\}$ implies $H_1(M) \neq \{0\}$. For a Riemann surface $H_1(M)$ is a free abelian group. Hence $Hom(\pi_1(M), \mathbf{Z}) \neq \{0\}$. □

LEMMA (CASE i). *Let M be a compact Riemann surface, $N \simeq \mathbf{P}_1$. Then for any holomorphic map $f: M \to N$ the Brouwer degree $deg(f)$ is non-negative.*

Proof. Holomorphic maps are orientation-preserving. □

LEMMA (CASE ii). *Let M be a compact Riemann surface of genus $g > 0$. Let N be a Riemann surface which is not simply-connected. Then there exists a continuous map from M to N which is not homotopic to a holomorphic one.*

Proof. Recall $H_1(M) \simeq \mathbf{Z}^{2g}$. $H_1(M)$ is the quotient of the fundamental group by its commutator group. It follows that there exists a group homomorphism $\rho: \pi_1(M) \to \pi_1(N)$ such that the image is a non-trivial cyclic subgroup C of $\pi_1(N)$. Let $N_1 \to N$ be the covering corresponding to $C \subset \pi_1(N)$. If there would exist a holomorphic map inducing ρ, then this holomorphic map $f: M \to N$ would be liftable to a map from M to N_1. However M is compact while N_1 is not (A compact Riemann surface can not have a non-trivial cyclic fundamental group). □

For the next case (iii) we need a classical result.

PROPOSITION 2. *A Riemann surface M admits a covering $\tau: A(r, 1) \to M$ with $0 < r < 1$ $(A(r, 1) = \{z \in \mathbf{C}: r < |z| < 1\})$ if and only if M is not isomorphic to $\mathbf{P}_1, \mathbf{C}, \mathbf{C}^*, \Delta, \Delta^*$ or a torus.*

Proof. It is easy to check that the listed Riemann surfaces do not admit such a covering. Hence let us assume that M is not isomorphic to one of those Riemann surfaces. By the uniformization theorem it follows that the universal covering $\tilde{M}$ is isomorphic to the unit disk. Hence $M = \Delta/\Gamma$ with $\Gamma \subset PSL_2(\mathbf{R})$. Consider the preimage $\Gamma_0 \subset SL_2(\mathbf{R})$. Every $\gamma \in \Gamma_0$ is conjugate in $SL_2(\mathbf{R})$ to either

$$t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ or } \theta = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \quad (\phi \in \mathbf{R})$$

$$\text{or } \zeta_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad (\lambda \in \mathbf{R}^*)\,.$$

However θ has necessarily a fixed point in Δ. Thus γ is conjugate to either t or ζ_λ. Let $\mathbf{Z}(t)$ be the generated group. Then $\Delta/\mathbf{Z}(t) \simeq \Delta^*$ and $\Delta/\mathbf{Z}(\zeta_\lambda) \simeq A(r, 1)$. Hence we have to prove that Γ_0 contains an element conjugate to a ζ_λ. Let us assume that there is no such element. Then every element in Γ_0 is conjugate to t and in particular its trace is 2. The elements of $SL_2(\mathbf{C})$ with trace 2 constitute an algebraic subvariety which contains the algebraic Zariski-closure of Γ_0 . It follows that the algebraic Zariski-closure is conjugate to the unipotent group U^+ of upper triangular matrices. Since $U^+ \cap SL_2(\mathbf{R}) \simeq \mathbf{R}$ it follows that Γ_0 is conjugate to the group of all elements of the form $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ with $n \in \mathbf{Z}$. But the quotient of Δ by this group is Δ^* which contradicts our assumptions. □

Remark. It should be noted that the above two equivalent conditions are furthermore equivalent to the property that M is hyperbolic and possesses a closed geodesic.

LEMMA 2. *Let M be a Riemann surface which is not simply-connected. Let $1 > r > 0$. Then there exists a continuous map $f: M \to A(r, 1)$ which is not homotopic to a holomorphic map.*

Proof. Assume the contrary. Let $\alpha \in H_1(M)$, $\alpha \neq 0$ and β the generator of $H_1(A(r, 1)) \simeq \mathbf{Z}$. Let f_n be holomorphic maps with $(f_n)_* \alpha = n\beta$. Regard the covering $\tau_n: A(\sqrt[n]{r}, 1) \to A(r, 1)$ given by $\tau_n(z) = z^n$. The maps f_n lift to maps g_n with $f_n = \tau_n \circ g_n$. The natural embedding $i: A(\sqrt[n]{r}, 1) \hookrightarrow A(r, 1)$ yields holomorphic maps $h_n = i \circ g_n$ with $h_n(M) \subset A(\sqrt[n]{r}, 1)$ and $(h_n)_* \alpha = \beta$. By the Montel theorem there is a convergent subsequence $\lim h_{n_k} = h$. Clearly $h_* \alpha = \beta$. On the other hand h must be constant, because holomorphic maps are open and the image of h is obviously contained in S^1 (recall $h_n(M) \subset A(\sqrt[n]{r}, 1)$). Thus we obtained a contradiction. □

Together with the preceding proposition this gives a proof for case iii).

LEMMA (CASE iv). *Let M_1 be a Riemann surface, $p \in M_1$ and $M = M_1 \setminus \{p\}$. Assume that M is not simply-connected. Then there exists a continuous map $g: M \to \Delta^*$ which is not homotopic to any holomorphic map.*

Proof. First consider the case where M_1 is compact. Recall that by the Riemann extension theorem any holomorphic map $f: M \to \Delta^*$ extends to a holomorphic map $\tilde{f}: M_1 \to \Delta$. If M_1 is compact, then f must be constant.

On the other side, there are continuous maps $g: N \to \Delta^*$ which are not homotopic to a constant map. This completes the proof if M_1 is compact. Now assume that M_1 is not compact. Let S be a contractible open neighbourhood of p with $S \setminus \{p\} \simeq \Delta^*$. A Maier-Vietoris-sequence implies that $H_1(S \setminus \{p\}) \to H_1(M)$ is injective. Hence any group homomorphism between the fundamental groups of $S \setminus \{p\}$ and Δ^* is induced by a continuous map $g: M \to \Delta^*$. On the other hand the Riemann extension theorem implies that any holomorphic map from $S \setminus \{p\}$ to Δ^* is homotopic to $z \mapsto z^k$ with $k \geqslant 0$. Thus a continuous map from M to Δ^* can not be homotopic to a holomorphic map in the case where the induced map $\partial S \to \partial \Delta$ is orientation reversing. □

LEMMA (CASE v). *Let M be a complex manifold for which $H_1(M)$ is a free abelian group with infinitely many generators. Then there exists a continuous map $f: M \to \Delta^*$ which is not homotopic to a holomorphic map.*

Proof. The homotopy classes of continuous maps are classified by $Hom(\pi_1(M), \mathbf{Z}) \simeq Hom(H_1(M), \mathbf{Z})$. Let γ_i denote the generators of $H_1(M)$ and β the generator of $H_1(\Delta^*)$. Any sequence $n_i \in \mathbf{Z}$ defines a group homomorphism by $\gamma_i \mapsto n_i \alpha$. Let $\tilde{M}$ denote the universal covering of M. We fix a base point in M resp. $\tilde{M}$. Let γ_i^0 be closed curves in M starting at the choosen base point and homologous to γ_i. These curves lift to curves in $\tilde{M}$ starting at the base point x_0 and ending at some points $p_i \in \tilde{M}$. Let d_i denote the Kobayashi-distance between x_0 and p_i. Assume that for given n_i there exists a holomorphic map $f: M \to \Delta^*$ inducing the corresponding group homomorphism. We may lift f to a map between the universal coverings $\tilde{f}: \tilde{M} \to H^+$ (as usual $H^+ = \{z: Im(z) > 0\}$). Then $\tilde{f}(p_i) = \tilde{f}(x_0) + n_i$ if the universal covering of Δ^* is given by $z \mapsto e^{2\pi i z}$. Now $d(\tilde{f}(x_0), \tilde{f}(x_0) + n_i) = \lambda n_i$ where λ is a positive number depending on $\tilde{f}(x_0)$. Since holomorphic maps are distance-decreasing we obtain $\lambda n_i \leqslant d_i$ for all i. Hence $0 < \lambda \leqslant d_i/n_i$ for all i. It follows that for n_i with $\inf_i(d_i/n_i) = 0$ there is no holomorphic map f with $f_* \gamma_i = n_i \alpha$. □

A VIEW TOWARD HIGHER DIMENSIONS

We will now investigate how to generalize our results to higher dimensions. First we want to give a generalization of the positive case 3).

Definition. Let G be a complex manifold. An open subset Ω is called an *attractive domain* if the injection map $i: \Omega \hookrightarrow G$ is a homotopy equivalence and furthermore for each compact subset $K \subset G$ there exists a holomorphic map $\phi_K: G \to G$ such that $\phi_K(K) \subset \Omega$ and furthermore ϕ_k is homotopy-equivalent to the identity map id_G.

THEOREM 4. *Let M be an open submanifold of a Stein manifold X such that M is relatively-compact and the injection map is a homotopy equivalence. Let Ω be an attractive domain in a complex-homogeneous manifold Y.*

Then every continuous map $f: M \to \Omega$ is homotopic to a holomorphic map $F: M \to \Omega$.

Proof. Since $M \hookrightarrow X$ is a homotopy-equivalence, there is a continuous map $f_1: X \to G$ homotopic to f.

By a result of Grauert and Kerner f_1 is homotopic to a holomorphic map $F_1: X \to G$. Now $F_1(\bar{M})$ is compact in G. Hence there exists $\phi: G \to G$ with $\phi \circ F_1(\bar{M}) \subset \Omega$. Thus $F = \phi \circ F_1|_M$ is the desired holomorphic map. □

Remark. (1) If M is a bounded pseudoconvex domain with smooth boundary in a Stein manifold X_0 then there exists a Stein manifold X with $M \subset X \subset X_0$ such that $M \hookrightarrow X$ is a homotopy equivalence. (Simply take a defining function f with $M = \{f < 0\}$ and let $X = \{f < \varepsilon\}$ for a sufficiently small $\varepsilon > 0$.)

(2) Let $G = (\mathbf{C}^*)^k$ and $K = (S^1)^k$. Let W be an open subset of $G/K \simeq \mathbf{R}^k$ such that the euclidean distance to the boundary $d(\cdot, \partial W)$ is an unbounded function on W. Then $\pi^{-1}(W)$ is an attractive domain in G, where π denotes the natural projection $G \to G/K$.

(3) The punctured unit ball $B_n \setminus \{(0, \ldots, 0)\}$ in $\mathbf{C}^n \setminus \{(0, \ldots, 0)\}$ is another example for an attractive domain.

Now we want to examine the negative case v). It reflects the general principle that hyperbolic manifolds (like Δ^*) are manifolds into which exist only few holomorphic maps. In particular there are too few maps to reflect a topology of "infinite type". Consider the following example: Let Λ be an infinite discrete subset of the disk Δ and consider holomorphic maps from $M = \Delta \setminus \Lambda$ to Δ^*. Any such map extends to a bounded holomorphic map on Δ. Now for $p \in \Lambda$ let $\gamma_p \in H_1(M)$ denote the corresponding cycle around p. Thus the Blaschke-condition for the zero sets of a bounded holomorphic function may be reformulated as follows: A group homomorphism $\phi: H_1(M) \to H_1(\Delta^*) \simeq \mathbf{Z}$ is induced by a holomorphic map if and only if

$\phi(\gamma_p) \geqslant 0$ for all $p \in \Lambda$ and $\sum_p \phi(\gamma_p) \cdot (1 - |p|) < \infty$. In particular it is impossible to choose the values $\phi(\gamma_p)$ independently for infinitely many γ_p's. This is a general fact as we will see below.

THEOREM 5. *Let X be a complex manifold and Y a complete hyperbolic manifold. Let Λ denote an infinite subset of $H_*(X)$. Then for any positive function $N: H_*(Y) \to \mathbf{R}^+$ there exists a positive function $\rho: \Lambda \to \mathbf{R}^+$ such that $\sum_{\gamma \in \Lambda} N(f_*\gamma) \cdot \rho(\gamma) < \infty$ holds for all $f \in Hol(X, Y)$.*

Remark. For simplicity let us assume that Λ is countable though in fact $H_*(X)$ itself is countable.

THEOREM 6. *Let X be a complex manifold, Y a complete hyperbolic manifold, $x_0 \in X$ and $K \subset Y$ a compact subset. Let H_K denote the set of all holomorphic maps $f: X \to Y$ with $f(x) \in K$. Let $\alpha \in H_*(X)$ be a homology class.*

Then $H_K(\alpha) = \{f_*\alpha : f \in H_K\}$ is finite.

For the proof we need the following lemma.

LEMMA 3. *Let Y be a differentiable manifold, K compact and $f_n: K \to Y$ a sequence of continuous maps uniformly converging to $f: K \to Y$. Then for some $N > 0$ all the maps f_n with $n > N$ are homotopic to f.*

Proof. We may endow Y with a Riemannian metric. Since $f(K)$ is compact, there is a number N such that for all $x \in K$ and $n > M$ $f_n(x)$ is contained in a normal neighbourhood of $f(x)$. Thus $f_n(x) = exp_{f(x)}(\mathbf{Z}_n(x))$ with $\mathbf{Z}_n(x) \in T_{f(x)}$. The desired homotopy is obtained by $f_{n,t}(x) = exp_{f(x)}(\mathbf{Z}_n(x))$ □

Proof of theorem 6. Since Y is complete hyperbolic, $Hol(X, Y)$ and H_K are normal families. From the compactness of K it follows that H_K can not contain a compactly divergent sequence. Hence any sequence $f_n \in H_K$ contains a subsequence converging uniformly on compact sets in X. Now every homology class α has compact support. Thus the preceding lemma implies that for every sequence $f_n \in H_K$ there is a subsequence f_{n_k} such that $f_{n_k *}\alpha$ becomes stationary. This is only possible, if $H_K(\alpha)$ is finite. □

Proof of theorem 5. Let γ_n $(n \in \mathbf{N})$ be an enumeration of Λ. Let K_n be an ascending sequence of compact subsets of Y with $\cup_n K_n = Y$. Let

$H_n = H_{K_n}$. Now $Hol(X, Y)$ is the union of all H_n. Now we choose $\rho(\gamma_n) \in \mathbf{R}^+$ in such a way that

$$(*) \qquad N(f_*(\gamma_n))\,\rho(\gamma_n) < 2^{-n}$$

for all $f \in H_n$. (This is possible by theorem 6.) Finally note that any holomorphic map $f \in Hol(X, Y)$ is contained in some H_N. Since $H_N \subset H_M$ for $N < M$ it follows that $(*)$ holds for all $n \geqslant N$. This clearly implies $\sum_n N(f_*\gamma_n)\,\rho(\gamma_n) < \infty$. □

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