5. Maximality of Mal'cev-Neumann fields

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each G). However, whereas the Witt functor is fully faithful on perfect fields of characteristic p, this new functor is not. For example, Proposition 11 (to be proved in Section 7) shows L can have many continuous (i.e. valuation-preserving) automorphisms not arising from automorphisms of R.

Our construction could be done starting from a non-abelian value group to produce p-adic Mal'cev-Neumann division rings, but we will not be interested in such objects.

5. Maximality of Mal'cev-Neumann fields

A valued field (E, w) is an *immediate extension* of another valued field (F, v) if

(1) E is a field extension of F, and $w|_F = v$.

(2) (E, w) and (F, v) have the same value groups and residue fields.

A valued field (F, v) is *maximally complete* if it has no immediate extensions other than (F, v) itself. (These definitions are due to F.K. Schmidt, but were first published by Krull [8].) For example, an easy argument shows that any field F with the trivial valuation, or with a discrete valuation making it complete, is maximally complete.

Proposition 6. *Let (F, v) be a maximally complete valued field with value group G and residue field R. Then*

(1) *F is complete.*

(2) *If R is algebraically closed and G is divisible, then F is algebraically closed.*

Proof. (1) The completion $\hat{F}$ of F is an immediate extension of F (see Proposition 5 in Chapter VI, §5, no. 3 of [2]), so $\hat{F} = F$.

(2) The algebraic closure $\bar{F}$ of F is in this case an immediate extension of F (see Proposition 6 in Chapter VI, §3, no. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]), so $\bar{F} = F$.

(This delightful trick is due to MacLane [10].) □

Proposition 7. *Any continuous endomorphism of a maximally complete field F which induces the identity on the residue field is automatically an automorphism (i.e., surjective).*

Proof. The field F is an immediate extension of the image of the endomorphism, which is maximally complete since it's isomorphic to F. □

From now on, when we refer to *Mal'cev-Neumann fields,* we mean one of the two fields K or L from the previous two sections. Let these have valuation v with value group G and residue field R. From now on, the proofs for the equal characteristic case K will be the same as (or easier than) those for the p-adic case L, so we will only give proofs for L. (To get a proof for K, simply replace p^g with t^g, and replace the set S of representatives with R.)

We will use the following lemma to show K and L are maximally complete.

LEMMA 4. *Let (F, v) be a valued field with value group G. Suppose we have an arbitrary system of inequalities of the form $v(x - a_\sigma) \geqslant g_\sigma$, with $a_\sigma \in F$ and $g_\sigma \in G$ for all σ in some index set I. Then*

(1) *If the system has a solution $x \in F$, then $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all $\sigma_1, \sigma_2 \in I$.*

(2) *Suppose in addition that $F = L$ (or K) is one of the Mal'cev-Neumann fields. Then the converse is true; i.e., if $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all $\sigma_1, \sigma_2 \in I$, then the system has a solution.*

Proof. (1) This is simply a consequence of the triangle inequality.

(2) Suppose $v(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all $\sigma_1, \sigma_2 \in I$. For each $g \in G$, let x_g be the coefficient of p^g in a_σ for any σ for which $g_\sigma > g$, and let $x_g = 0$ if no such σ exists. We claim x_g is uniquely defined. For if $g_{\sigma_1}, g_{\sigma_2} > g$, then $v(a_{\sigma_1} - a_{\sigma_2}) > g$, so by Lemma 3 the coefficients of p^g in $a_{\sigma_1}, a_{\sigma_2}$ must be the same.

Define $x = \sum_{g \in G} x_g p^g$. To show $x \in L$, we must check that $\operatorname{Supp} x$ is well-ordered. Suppose $h_1, h_2, \ldots$ is a strictly descending sequence within $\operatorname{Supp} x$. Then by definition of x_g, $h_1 < g_\sigma$ for some $\sigma \in I$, and $h_n \in \operatorname{Supp} a_\sigma$ for all $n \geqslant 1$. This is a contradiction, since $\operatorname{Supp} a_\sigma$ is well-ordered. Thus $x \in L$.

By definition of x_g, the coefficients of p^g in x and a_σ agree for $g < g_\sigma$. From Lemma 3 it follows that $v(x - a_\sigma) \geqslant g_\sigma$. $\square$

THEOREM 1 (Krull [8]). *The Mal'cev-Neumann fields K and L are maximally complete. (Actually, Krull proved this only for the equal characteristic case (K), but his proof applies equally well to the p-adic fields L.)*

Proof. (As usual, we treat only the p-adic case.) Suppose (M, w) is a proper immediate extension of (L, v). Fix $\mu \in M \backslash L$. Consider the system of inequalities $w(x - a_\sigma) \geqslant g_\sigma$, where a_σ ranges over all elements of L and $g_\sigma = w(\mu - a_\sigma)$. Obviously μ is a solution (in M), so by part 1 of Lemma 4,

$w(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ for all σ_1, σ_2. Now $v(a_{\sigma_1} - a_{\sigma_2}) = w(a_{\sigma_1} - a_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$, so we may apply part 2 of Lemma 4 to deduce that the system of inequalities $v(x - a_\sigma) \geqslant g_\sigma$ has a solution $\lambda \in L$.

The idea is that λ is a best approximation in L to μ. We will contradict this by adding the "leading term" of the difference $\mu - \lambda$ to λ to get a better one. Since $\mu \notin L$, $\mu - \lambda \neq 0$, so we can let $g = w(\mu - \lambda) \in G$. (Here we are using that L and M have the same value group.) Then $w(p^{-g}(\mu - \lambda)) = 0$, so there exists a unique representative $s \in S$ for the (nonzero) residue class containing $p^{-g}(\mu - \lambda)$. (Here we are using that L and M have the same residue field.) Then $w(p^{-g}(\mu - \lambda) - s) > 0$, so $w(\mu - \lambda - sp^g) > g$. On the other hand, $g = v(-sp^g) = v(\lambda - (\lambda + sp^g)) \geqslant w(\mu - (\lambda + sp^g))$, by the definition of λ, using $a_\sigma = \lambda + sp^g$. This contradiction proves L is maximally complete. $\square$

Remark. It is true in general that F is maximally complete iff part 2 of Lemma 4 is true for F. See Kaplansky's discussion of pseudolimits [5], and Theorem 5 in Chapter I of [4].

COROLLARY 4. *Any Mal'cev-Neumann field is complete. A Mal'cev-Neumann field with divisible value group and algebraically closed residue field is itself algebraically closed.*

Proof. Combine the previous theorem with Proposition 6. $\square$

Remark. In practice, to find solutions to a polynomial equation over a Mal'cev-Neumann field, one can use successive approximation. This method could be used to give another (much messier) proof that these Mal'cev-Neumann fields are algebraically closed.

We will show that the Mal'cev-Neumann fields K and L are maximal in a sense much stronger than Theorem 1 implies. This will be made precise in Corollary 5.

THEOREM 2. *Suppose L (or K) is a Mal'cev-Neumann field with valuation v having divisible value group G and algebraically closed residue field R. Suppose E is a subfield of L, and that (F, w) is a valued field extension of (E, v), with value group contained in G and residue field contained in R. Then there exists an embedding of valued fields $\phi: F \to L$ which extends the inclusion $E \hookrightarrow L$.*

Proof. Since G is divisible and R is algebraically closed, we can extend the valuation on F to a valuation on $\bar{F}$ with value group in G and residue field in R, by Proposition 6 in Chapter VI, §3, No. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]. If we could find an embedding of $\bar{F}$ into L, we would get an embedding of F into L. Thus we may assume that F is algebraically closed.

Let $\mathscr{C}$ be the collection of pairs (E', ϕ) such that E' is a field between E and F and $\phi: E' \to L$ is an embedding of valued fields. Define a partial order on $\mathscr{C}$ by saying (E_2', ϕ_2) is above (E_1', ϕ_1) if $E_2' \supseteq E_1'$ and ϕ_2 extends ϕ_1. By Zorn's Lemma, we can find a maximal element (E', ϕ) of $\mathscr{C}$. By relabeling elements, we can assume $E' \subseteq L$, and we may as well rename E' as E.

We claim this E is algebraically closed. Both F and L are algebraically closed. (For L, this follows from Corollary 4.) So we have an algebraic closure of E in F and in L, each with a valuation extending the valuation on E. By Corollary 1 in Chapter VI, §8, No. 6 of [2], two such valuations can differ only by an automorphism of $\bar{E}$ over E; i.e., there exists a continuous embedding of the algebraic closure of E in F into L. By maximality of (E, ϕ) in $\mathscr{C}$, E must be algebraically closed already.

If $E = F$, we are done, so assume there is some element $\mu \in F \backslash E$. We will define a corresponding element $\mu' \in L$.

Case 1: There exists a best approximation $e_0 \in E$ to μ; i.e. there exists $e_0 \in E$ such that $w(\mu - e) \leqslant w(\mu - e_0)$ for all $e \in E$. Let $g = w(\mu - e_0) \in G$.

Case 1a: $g \notin v(E)$. Then define $\mu' = e_0 + p^g$.

Case 1b: $g = v(\delta)$ for some $\delta \in E$. Then $w(\delta^{-1}(\mu - e_0)) = 0$, so we let $s \in S$ be the representative of the (nonzero) residue class corresponding to $\delta^{-1}(\mu - e_0) \in F$, and define $\mu' = e_0 + s\delta$.

Note that in these cases, $v(\mu' - e_0) = g$, so for all $e \in E$,

$$\begin{aligned}
v(\mu' - e) &\geqslant \min\{v(\mu' - e_0), v(e - e_0)\} \quad \text{(the triangle inequality)}\\
&= \min\{g, v(e - e_0)\}\\
&= \min\{w(\mu - e_0), w(e - e_0)\} \quad \text{(since } v \text{ and } w \text{ agree on } E)\\
&\geqslant \min\{w(\mu - e_0), w(\mu - e), w(\mu - e_0)\} \quad \text{(the triangle inequality)}\\
&= w(\mu - e) \quad \text{(by definition of } e_0).
\end{aligned}$$

Case 2: For every $e \in E$, there exists $e' \in E$ with $w(\mu - e') > w(\mu - e)$.

Consider the system of inequalities $w(x - e_\sigma) \geqslant g_\sigma$, where e_σ ranges over all elements of E and $g_\sigma = w(\mu - e_\sigma)$. Since μ is a solution (in F), $w(e_{\sigma_1} - e_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\}$ by part 1 of Lemma 4. We have

$$v(e_{\sigma_1} - e_{\sigma_2}) = w(e_{\sigma_1} - e_{\sigma_2}) \geqslant \min\{g_{\sigma_1}, g_{\sigma_2}\},$$

so by part 2 of Lemma 4, the system of inequalities $\upsilon(x - e_\sigma) \geqslant g_\sigma$ has a solution μ' in L.

Claim. In all cases, $w(\mu - e) = \upsilon(\mu' - e)$ for all $e \in E$.

Proof. From the remarks at the end of Case 1, and by the definition of μ' in Case 2, we have $w(\mu - e) \leqslant \upsilon(\mu' - e)$ for all $e \in E$.

First suppose e is not a best approximation to μ, so $w(\mu - e') > w(\mu - e)$, for some $e' \in E$. Then equality holds in the triangle inequality,

$$w(e - e') = w((\mu - e') - (\mu - e)) = w(\mu - e)$$

so

$$\upsilon(e - e') = w(e - e') = w(\mu - e) < w(\mu - e') \leqslant \upsilon(\mu' - e') .$$

Again equality holds in the triangle inequality, so we get

$$\upsilon(\mu' - e) = \upsilon((\mu' - e') - (e - e')) = \upsilon(e - e') = w(\mu - e)$$

which proves the claim in this case.

Thus we are left with the case in which $w(\mu - e') \leqslant w(\mu - e)$ for all $e' \in E$. Then Case 1 holds and $w(\mu - e) = w(\mu - e_0) = g$. Suppose $\upsilon(\mu' - e) > g$. Then applying the triangle equality to $e - e_0 = (\mu' - e_0) - (\mu' - e)$ and using $\upsilon(\mu' - e_0)$ from our remarks at the end of Case 1, we get $\upsilon(e - e_0) = \upsilon(\mu' - e_0) = g$. Thus $g \in \upsilon(E)$ so we must be in Case 1b. Moreover

$$\upsilon(\delta^{-1}(\mu' - e_0) - \delta^{-1}(e - e_0)) = \upsilon(\delta^{-1}) + \upsilon(\mu' - e) > -g + g = 0$$

so $\delta^{-1}(\mu' - e_0)$ and $\delta^{-1}(e - e_0)$ have the same image in the residue field R. But by definition of μ' in Case 1b, $\delta^{-1}(\mu' - e_0)$ has the same image in R as $\delta^{-1}(\mu - e_0)$. Combining these facts gives us

$$w(\delta^{-1}(\mu - e_0) - \delta^{-1}(e - e_0)) > 0$$

so $w(\mu - e) > w(\delta) = \upsilon(\delta) = g$, contradicting the definitions of g and e_0. Thus we cannot have $\upsilon(\mu' - e) > g$. But we know $\upsilon(\mu' - e) \geqslant w(\mu - e) = g$, so we must have $\upsilon(\mu' - e) = w(\mu - e) = g$. This completes the proof of the claim.

Since $\mu \notin E$, $\upsilon(\mu' - e) = w(\mu - e) \neq \infty$ for all $e \in E$. Hence $\mu' \notin E$. But E is algebraically closed, so μ and μ' are transcendental over E, and we have an isomorphism of fields $\Phi: E(\mu) \to E(\mu')$ over E which maps μ to μ'.

We claim that Φ preserves the valuation. (The valuations on $E(\mu)$, $E(\mu')$ are the restrictions of w, υ respectively). Since E is algebraically closed , any element $\rho \in E(\mu)$ can be written

$$\rho = \varepsilon_0(\mu - \varepsilon_1)^{n_1}(\mu - \varepsilon_2)^{n_2} \cdots (\mu - \varepsilon_k)^{n_k} ,$$

for some $\varepsilon \in E$ and $n_i \in \mathbf{Z}$. By the Claim above, and the fact that v and w agree on E, it follows that $w(\rho) = v(\Phi(\rho))$, as desired.

But $(E(\mu), \Phi)$ contradicts the maximality of (E, ϕ) in $\mathscr{C}$. Thus we must have had $E = F$, so we are done. $\square$

COROLLARY 5. *Let (F, v) be a valued field with value group contained in a divisible ordered group G, and residue field contained in an algebraically closed field R. Define K and L as usual as the Mal'cev-Neumann fields with value group G and residue field R. (Define the p-adic Mal'cev-Neumann field L only if* $\operatorname{char} R > 0$.*) Then there exists an embedding of valued fields $\phi: F \to K$ or $\phi: F \to L$, depending on if the restriction of v to the minimal subfield of F is the trivial valuation (on $\mathbf{Q}$ or $\mathbf{F}_p$) or the p-adic valuation on $\mathbf{Q}$.*

Proof. Apply Theorem 2 with E as the minimal subfield. $\square$

COROLLARY 6. *Every valued field F has at least one immediate extension which is maximally complete. If the value group G is divisible and the residue field R is algebraically closed, then there is only one (up to isomorphism).*

Proof. Embed F in a Mal'cev-Neumann field L (or K) with value group $\tilde{G}$ and residue field $\bar{R}$, according to the previous corollary. Let $\mathscr{C}$ be the collection of valued subfields of L which are immediate extensions of F. By Zorn's Lemma, $\mathscr{C}$ has a maximal element M. If M had an immediate extension M', then by Theorem 2, we could embed M' in L. This would contradict the maximality of M.

If G is divisible and the R is algebraically closed, then any maximally complete immediate extension M of F can be embedded in L, and L is an immediate extension of M, so $L = M$. $\square$

Remarks. Krull [8] was the first to prove that every valued field F had a maximal extension. His proof involves showing directly that there exists a bound on the cardinality of a valued field with given value group and residue field. Then Zorn's Lemma is applied.

Kaplansky [5] has investigated in detail the question of when the maximally complete immediate extension is unique. He has found weaker conditions on the value group and residue field which guarantee this extension is unique. If $\operatorname{char} R = 0$, the extension is unique. If $\operatorname{char} R = p > 0$, the extension is unique if the following pair of conditions is satisfied:

(1) Any equation of the form

$$x^{p^n} + a_1 x^{p^{n-1}} + \cdots + a_{n-1}x^p + a_n x + a_{n+1} = 0$$

with coefficients in R has a root in R.

(2) The value group G satisfies $G = pG$.

Also if G is discrete of arbitrary rank and char F = char R, then the extension in unique [6]. But Kaplansky gives examples where the extension is not unique. The exact conditions under which the extension is unique are not known.

6. Applications

One application of Theorem 2 is to the problem of "glueing" two valued fields. (This result can also be proved directly without the use of Mal'cev-Neumann fields; it is equivalent to Exercise 2 for §2 in Chapter VI of [2]. Our method has the advantage of showing that the value group of the composite field can be contained in any divisible value group large enough to contain the value groups of the fields to be glued.)

Proposition 8. *Suppose* E, F, F' *are valued fields and that we are given embeddings of valued fields* $\phi : E \to F, \phi' : E \to F'$. *Then there exist a Mal'cev-Neumann field* L *(or* K*) and embeddings of valued fields* $\Phi : F \to L, \Phi' : F' \to L$ *such that* $\Phi \circ \phi = \Phi' \circ \phi'$.

Proof. By the glueing theorem for ordered groups [14], we can assume the value groups of F and F' are contained in a single ordered group G. Also we can assume that their residue fields are contained in a field R. Moreover, we may assume G is divisible and R is algebraically closed. Then E can be embedded as a valued subfield of a power series field L (or K) with value group G and residue field R, by Corollary 5. Finally, Theorem 2 gives us the desired embeddings Φ, Φ'. $\square$

Remark. Transfinite induction can be used to prove the analogous result for glueing an arbitrary collection of valued fields.

Since a non-archimedean absolute value on a field can be interpreted as a valuation with value group contained in $\mathbf{R}$, we can specialize the results of Section 5 to get results about fields with non-archimedean absolute values. For example, Corollary 5 implies the following, which may be considered the non-archimedean analogue of Ostrowski's theorem that any field with an archimedean absolute value can be embedded in $\mathbf{C}$ with its usual absolute value (or one equivalent).