4. p-ADIC MAL'CEV-NEUMANN FIELDS

To construct analogous examples of characteristic zero whose residue field has nonzero characteristic requires a more complicated construction. First we recall two results about complete discrete valuation rings. For proofs, see [17], pp. 32-34.

A valued field (F, v) is called *discrete* if $v(F) = \mathbf{Z}$.

PROPOSITION 1. *If* R *is a perfect field of characteristic* $p > 0$, *then there exists a unique field* R' *of characteristic* 0 *with a discrete valuation* v *such that the residue field is* R, $v(p) = 1 \in \mathbf{Z}$, *and* R' *is complete with respect to* v. *(The valuation ring* A *of* R' *is called the ring of* Witt vectors *with coefficients in* R.*)*

For example, if $R = \mathbf{F}_p$, then $R' = \mathbf{Q}_p$ with the p-adic valuation.

PROPOSITION 2. *Suppose* F *is field with a discrete valuation* v, *and* $t \in F$ *satisfies* $v(t) = 1$. *Let* $S \subset F$ *be a set of representatives for the residue classes with* $0 \in S$. *Then every element* $x \in F$ *can be written uniquely as* $\sum_{m \in \mathbf{Z}} x_m t^m$, *where* $x_m \in S$ *for each* m, *and* $x_m = 0$ *for all sufficiently negative* m. *Conversely, if* F *is complete, every such series defines an element of* F.

Now for the construction. Let R be a perfect field of characteristic p, and let G be an ordered group containing $\mathbf{Z}$ as a subgroup, or equivalently with a distinguished positive element. (When we eventually define our valuation v, this element $1 \in G$ will be $v(p)$.) Let A be the valuation ring of the valued field (R', v') given by Proposition 1.

What we want is to have the indeterminate t stand for p in elements of $A((G))$, so we get elements of the form $\sum_{g \in G} \alpha_g p^g$. The problem is that some elements of $A((G))$, like $-p + t^1$, "should be" zero. So what we do is to take a quotient $A((G))/N$ where $N \subset A((G))$ is the ideal of elements that "should be" zero.

We say that $\alpha = \sum_g \alpha_g t^g \in A((G))$ is a *null series* if for all $g \in G$, $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = 0$ in R'. (Recall that we fixed a copy of $\mathbf{Z}$ in G.) Note that $\alpha_{g+n} = 0$ for sufficiently negative n, since otherwise $\operatorname{Supp} \alpha$ would not be well-ordered. Also, $v'(\alpha_{g+n} p^n) \geqslant n$, so $\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n$ always converges in R'. Let N be the set of null series.

PROPOSITION 3. *N is an ideal of $A((G))$.*

Proof. Clearly N is an additive subgroup. Let $G' \subset G$ be a set of coset representatives for $G/\mathbf{Z}$. Suppose $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, $\beta = \sum_{h \in G} \beta_h t^h \in N$, and $\alpha\beta = \sum_{j \in G} \gamma_j t^j$. Then for each $j \in G$,

$$\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = \sum_{\substack{g+h=j+n \\ n \in \mathbf{Z}}} \alpha_g \beta_h p^n$$

$$= \sum_{\substack{h' \in G' \\ l, m \in \mathbf{Z}}} (\alpha_{j-h'+l} p^l)(\beta_{h'+m} p^m)$$

(We write $h = h' + m$ with $h' \in G'$ and let $l = n - m$.)

Since $\beta \in N$, $\sum_{m \in \mathbf{Z}} \beta_{h'+m} p^m = 0$ for each $h' \in G$, so we get $\sum_{n \in \mathbf{Z}} \gamma_{j+n} p^n = 0$. (These infinite series manipulations in R' are valid, because for each $i \in \mathbf{Z}$, only finitely many terms have valuation less than i, since each γ_{j+n} is a finite sum of products $\alpha_g \beta_h$.) Hence N is an ideal. □

Define the p-adic Mal'cev-Neumann field L as $A((G))/N$.

PROPOSITION 4. *Let $S \subset A$ be a set of representatives for the residue classes of A, with $0 \in S$. Then any element $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$ is equivalent modulo N to a element $\beta = \sum_{g \in G} \beta_g t^g$ with each β_g in S. Moreover, β is unique.*

Proof. Let $G' \subset G$ be a set of coset representatives for $G/\mathbf{Z}$. For each $g \in G'$, we may write

$$\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n = \sum_{n \in \mathbf{Z}} \beta_{g+n} p^n$$

with $\beta_{g+n} \in S$, by Proposition 2. (This is possible since R' is complete with respect to its discrete valuation.) Then $\beta = \sum_{g \in G'} \sum_{n \in \mathbf{Z}} \beta_{g+n} t^n$ is a well-defined element of $A((G))$, since $\operatorname{Supp}(\beta) \subseteq (\operatorname{Supp} \alpha) + \mathbf{N}$, which is well-ordered by part 2 of Lemma 1. Finally $\alpha - \beta \in N$, by definition of N. The uniqueness follows from the uniqueness in Proposition 2. □

COROLLARY 3. *$L = A((G))/N$ is a field.*

Proof. The previous proposition shows that any $\alpha \in A((G))$ is equivalent modulo N to 0 or an element which is a unit in $A((G))$ by Corollary 1. □

Proposition 4 allows us to write an element of L uniquely (and somewhat carelessly) as $\beta = \sum_{g \in G} \beta_g p^g$, with $\beta_g \in S$. Thus given S, we can speak of $\operatorname{Supp}(\beta)$ for $\beta \in L$. Define $v : L \to G_\infty$ by $v(\beta) = \min \operatorname{Supp} \beta$.

PROPOSITION 5. *The map* v *is a valuation on* L, *and is independent of the choice of* S. *The value group is* G *and the residue field is* R.

Proof. For $\alpha = \sum_{g \in G} \alpha_g t^g \in A((G))$, define

$$w(\alpha) = \min_{g \in G} \left\{ g + v'\Big(\sum_{n \in \mathbf{Z}} \alpha_{g+n} p^n\Big) \right\}.$$

The elements in the "min" belong to $(\operatorname{Supp}(\alpha) + \mathbf{N}) \cup \{\infty\}$, which is well-ordered by part 2 of Lemma 1, so this is well defined. It's clearly unchanged if an element of N is added to α. In particular, if we do so to get an element $\alpha' \in A((G))$ with coefficients in S, we find $w(\alpha) = w(\alpha') = \min \operatorname{Supp} \alpha'$. Thus if β is the image of α in L, $v(\alpha) = w(\beta)$. Since w is independent of the choice of S, so is v. If α', β' are the representatives in $A((G))$ with coefficients in S of elements $\alpha, \beta \in L$, then it is clear that $w(\alpha'\beta') = w(\alpha') + w(\beta')$ (because the leading coefficient of $\alpha'\beta'$ has valuation 0 under v') and that $w(\alpha' + \beta') \geqslant \min\{w(\alpha'), w(\beta')\}$. Thus v is a valuation.

The value group of v is all of G, since $v(p^g) = g$ for any $g \in G$. The natural inclusion $A \subset A((G))$ composed with the quotient map $A((G)) \to L$ maps A into the valuation ring of L, which consists of series $\sum_{g \geqslant 0} \alpha_g p^g$, so it induces a map ϕ from A to the residue field of L. The residue class of $\sum_{g \geqslant 0} \alpha_g p^g$ equals $\phi(\alpha_0) \in A$ (since the maximal ideal for L consists of series $\sum_{g > 0} \alpha_g p^g$). Thus ϕ is surjective. Its kernel is the maximal ideal of A, so ϕ induces an isomorphism from the residue class field of A to that of L. □

For example, if R is any perfect field of characteristic p, and $G = k^{-1}\mathbf{Z}$ for some $k \geqslant 1$ (with its copy of $\mathbf{Z}$ as a subgroup of index k), then $L = R'(\sqrt[k]{p})$ with the p-adic valuation.

LEMMA 3. *If* $\alpha = \sum_{g \in G} \alpha_g p^g$ *and* $\beta = \sum_{g \in G} \beta_g p^g$ *with* $\alpha_g, \beta_g \in S$ *are two elements of* L, *then* $v(\alpha - \beta) = \min\{g \in G \mid \alpha_g \neq \beta_g\}$. *(The corresponding fact for the usual Mal'cev-Neumann fields is obvious.)*

Proof. Let w be the map used in the proof of the previous proposition. Let $\alpha' = \sum_{g \in G} \alpha_g t^g$ and $\beta' = \sum_{g \in G} \beta_g t^g$ in $A((G))$. Then $v(\alpha - \beta) = w(\alpha' - \beta')$. If $g_0 = \min\{g \in G \mid \alpha_g \neq \beta_g\}$, then the leading term of $\alpha' - \beta'$ is $(\alpha_{g_0} - \beta_{g_0}) t^{g_0}$, and the leading coefficient here has valuation 0 under v', since $\alpha_{g_0}, \beta_{g_0}$ represent distinct residue classes, so $w(\alpha' - \beta') = g_0$, as desired. □

Remarks. Since the construction of A from R is functorial (the Witt functor), it is clear that the construction of L from R is functorial as well (for each G). However, whereas the Witt functor is fully faithful on perfect fields of characteristic p, this new functor is not. For example, Proposition 11 (to be proved in Section 7) shows L can have many continuous (i.e. valuation-preserving) automorphisms not arising from automorphisms of R.

Our construction could be done starting from a non-abelian value group to produce p-adic Mal'cev-Neumann division rings, but we will not be interested in such objects.

5. Maximality of Mal'cev-Neumann fields

A valued field (E, w) is an *immediate extension* of another valued field (F, v) if

(1) E is a field extension of F, and $w|_F = v$.

(2) (E, w) and (F, v) have the same value groups and residue fields.

A valued field (F, v) is *maximally complete* if it has no immediate extensions other than (F, v) itself. (These definitions are due to F.K. Schmidt, but were first published by Krull [8].) For example, an easy argument shows that any field F with the trivial valuation, or with a discrete valuation making it complete, is maximally complete.

Proposition 6. *Let (F, v) be a maximally complete valued field with value group G and residue field R. Then*

(1) *F is complete.*

(2) *If R is algebraically closed and G is divisible, then F is algebraically closed.*

Proof. (1) The completion $\hat{F}$ of F is an immediate extension of F (see Proposition 5 in Chapter VI, §5, no. 3 of [2]), so $\hat{F} = F$.

(2) The algebraic closure $\bar{F}$ of F is in this case an immediate extension of F (see Proposition 6 in Chapter VI, §3, no. 3 and Proposition 1 in Chapter VI, §8, no. 1 of [2]), so $\bar{F} = F$.

(This delightful trick is due to MacLane [10].) □

Proposition 7. *Any continuous endomorphism of a maximally complete field F which induces the identity on the residue field is automatically an automorphism (i.e., surjective).*

Proof. The field F is an immediate extension of the image of the endomorphism, which is maximally complete since it's isomorphic to F. □