

## 5. $\bar{W}$ IS PATH CONNECTED

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5.  $\bar{W}$  IS PATH CONNECTED

Here we refine the argument of the previous section to prove  $\bar{W}$  is path connected. There are two main difficulties that arise. One is that the path connected analogue of Lemma 4.2, although still true (at least when  $M$  is Hausdorff), is much harder to prove. The second is that a decreasing intersection of compact path connected sets need not be path connected, so we can no longer restrict our attention to the zeros within  $\{z: |z| < 1 - \delta\}$ .

The lifting lemma below will be used as a substitute for Lemma 4.2. Its proof is based on proofs obtained independently by David desJardins and Emanuel Knill.

**LEMMA 5.1.** (Lifting lemma): *Let  $M$  be a Hausdorff space and let  $\pi: M^n \rightarrow M^n/S_n$  be the projection map. Let  $f: [0, 1] \rightarrow M^n/S_n$  be a continuous map. Then there is a continuous map  $g: [0, 1] \rightarrow M^n$  such that  $f = \pi \circ g$ .*

**SUBLEMMA 5.1.** *Let  $\Delta = \{t \in [0, 1]: f(t) \text{ consists of } n \text{ copies of a single point}\}$ . Let  $g: [0, 1] \rightarrow M^n$  be an arbitrary function that is a lift of  $f$ . Then  $g$  is automatically continuous at all  $t_0 \in \Delta$ .*

*Proof.* Suppose  $t_0 \in \Delta$  and  $f(t_0) = \{x, x, \dots, x\}$ . If  $U$  is an open neighborhood of  $x$ ,

$$g^{-1}(U^n) = f^{-1}(\pi(U^n))$$

which is open. Since such subsets  $U^n$  form a neighborhood base at  $(x, x, \dots, x) \in M^n$ , this proves that  $g$  is continuous at  $t_0$ .

**SUBLEMMA 5.2.** *Let  $I_1, I_2$  be closed subintervals of  $[0, 1]$  such that  $I_1 \cap I_2$  is a single point  $\{t\}$ . If  $g_j$  is a continuous lift of  $f$  on  $I_j$  ( $j = 1, 2$ ) then there is a continuous lift  $g$  of  $f$  on  $I_1 \cup I_2$  such that  $g|_{I_1} = g_1$ .*

*Proof.* Since  $g_1(t)$  and  $g_2(t)$  differ only by a permutation, we can compose  $g_2$  with a permutation  $\sigma: M^n \rightarrow M^n$  and then paste the result to  $g_1$ .

**SUBLEMMA 5.3.** *The conclusions of Sublemma 5.2 hold even if  $I_1$  and  $I_2$  intersect in more than a point.*

*Proof.* This follows from Sublemma 5.2 since  $I_1 \cup I_2$  can be expressed as the union of  $I_1$  with at most two closed subintervals of  $I_2$  each meeting  $I_1$  in a point.

**SUBLEMMA 5.4.** *If  $I$  is a closed subinterval of  $[0, 1]$  and every  $t \in I$  has a neighborhood on which  $f$  has a lift, then  $f$  has a lift on  $I$ .*

*Proof.* By compactness, we can cover  $I$  by closed intervals  $I_1, I_2, \dots, I_k$  on which  $f$  has a lift, and we may assume  $I_j \cap I_{j+1} \neq \emptyset$  for  $1 \leq j \leq k$ . By induction on  $j$ , Sublemma 5.3 lets us extend the lift on  $I_1$  to a lift on  $I_1 \cup I_2 \cup \dots \cup I_j$ .

**SUBLEMMA 5.5.** *The same holds if  $I$  is any subinterval of  $[0, 1]$ .*

*Proof.* Let  $C_1 \subseteq C_2 \subseteq \dots$  be closed intervals such that  $\bigcup_{i=1}^{\infty} C_i = I$ . By Sublemma 5.4, there is a lift on each  $C_i$ . By repeated use of Sublemma 5.3, extend the lift on  $C_1$  to a lift on  $C_2$ , extend this to  $C_3$ , etc. This process gives a lift on  $I$ .

*Proof of Lemma 5.1.* We use induction on  $n$ . The case  $n = 1$  is trivial, so assume  $n > 1$ . By Sublemma 5.1, it suffices to find a lift on each connected component  $I$  of  $[0, 1] \setminus \Delta$ . By Sublemma 5.5 it suffices to show that any  $t_0 \in I$  has a neighborhood on which there is a lift.

Suppose  $z_1, z_2, \dots, z_k$  ( $k \geq 2$ ) are the distinct elements of the multiset  $f(t_0)$ , occurring with multiplicities  $n_1, n_2, \dots, n_k$  respectively. Since  $M$  is Hausdorff, there exist pairwise disjoint neighborhoods  $U_i$  of  $z_i$ . Let  $N$  be a closed interval neighborhood of  $t_0$  such that  $t \in N$  implies  $f(t) \in \pi(U_1^{n_1} \times \dots \times U_k^{n_k})$ . Then on  $N$ , we can lift  $f$  to a path  $\tilde{f}$  in  $M^{n_1}/S_{n_1} \times \dots \times M^{n_k}/S_{n_k}$  since the projection

$$M^{n_1}/S_{n_1} \times \dots \times M^{n_k}/S_{n_k} \rightarrow M^n/S_n$$

restricts to a homeomorphism on the projections of  $U_1^{n_1} \times \dots \times U_k^{n_k}$ . By the inductive hypothesis applied to each of the  $k$  coordinates of  $\tilde{f}$ , we can lift  $\tilde{f}$  to a path in  $M^{n_1} \times \dots \times M^{n_k} = M^n$  as desired.  $\square$

**THEOREM 5.1.**  $\bar{W}$  is path connected.

*Proof.* Let  $M$  be  $\{z: |z| \leq 1\}$  with the unit circle shrunk to a point  $P$ . Again  $M$  is topologically a sphere, so we may give it a bounded metric  $d$ .

Let  $M^\infty$  be the set of sequences  $x = \{x_i\}_{i=1}^\infty$  which converge to  $P$  and define a metric  $d_\infty$  on  $M^\infty$  by

$$d_\infty(x, y) = \sup_i d(x_i, y_i) .$$

Let the group  $S_\infty$  of permutations of  $\{1, 2, \dots\}$  act on  $M^\infty$  by permuting the coordinates. Define a metric  $D$  on the quotient space  $M^\infty/S_\infty$  by letting

$$D(\bar{x}, \bar{y}) = \inf_{\sigma \in S_\infty} d_\infty(x, \sigma y) .$$

Here  $(\bar{x}, \bar{y})$  denote the projections of  $x, y \in M^\infty$  to  $M^\infty/S_\infty$ . (That  $D(\bar{x}, \bar{y}) = 0$  if and only if  $\bar{x} = \bar{y}$  requires the convergence of  $x, y$ .) The set of zeros of a power series

$$1 + \varepsilon_1 z + \varepsilon_2 z^2 + \dots$$

inside  $\{z: |z| < 1\}$  forms a sequence in  $M$  converging to  $P$  (by Proposition 2.1) or else is finite, in which case we append an infinite sequence of  $P$ 's. This defines a map

$$f: \{0, 1\}^\omega \rightarrow M^\infty/S_\infty .$$

By the same Rouché's theorem argument used in the proof of Theorem 4.1, this map is continuous. The conditions of Lemma 4.1 hold for the same reason as before, so the image of  $f$  is path connected.

Suppose  $z_0 \in \bar{W} \cap \{z: |z| < 1\}$ . Let  $\omega: [0, 1] \rightarrow M^\infty/S_\infty$  be a path from the image under  $f$  of a 0, 1 power series vanishing at  $z_0$  to  $f((0, 0, \dots)) = \{P, P, P, \dots\}$ .

Fix  $m \geq 1$ , and let  $M_m$  be  $\{z: |z| \leq 1\}$  with the annulus

$$\{z: 1 - 1/m \leq |z| \leq 1\}$$

shrunk to a point  $Q$ . Define  $||$  on  $M_m$  by letting  $|Q| = 1 - 1/m$ . By Proposition 2.1 there is an upper bound  $n$  on the number of zeros of a 0, 1 power series inside  $\{z: |z| < 1 - 1/m\}$ . The path  $\omega$  induces a path

$$\omega_m: [0, 1] \rightarrow (M_m)^n/S_n .$$

(Apply the projection  $M \rightarrow M_m$  to each element of  $\omega(t)$ , and throw away infinitely many  $Q$ 's to get  $\omega_m(t)$ .)

Pick  $m_0 \geq 1$  such that  $|z_0| < 1 - 2/m_0$ . We define inductively a sequence of paths

$$\tilde{\omega}_m: [0, t_m] \rightarrow \bar{W} , \quad m = m_0, m_0 + 1, \dots ,$$

each extending the one before. First apply Lemma 5.1 to lift  $\omega_{m_0}$  to a path  $[0, 1] \rightarrow M_{m_0}^n$ . Since some coordinate of  $\omega_{m_0}(0)$  is  $z_0$  and since all coordinates of  $\omega_{m_0}(1)$  are  $Q$ , we get a path  $\tilde{\omega}_m$  from  $z_0$  to  $Q$  in  $M_{m_0}$ . Let  $t_{m_0}$  be the smallest  $t \in [0, 1]$  such that  $|\tilde{\omega}_{m_0}(t)| \geq 1 - 2/m_0$ . Then by restriction to  $[0, t_{m_0}]$  we get a path  $\tilde{\omega}_{m_0}$  in  $\mathbb{C}$  since  $\{z \in M_{m_0} : |z| \leq 1 - 2/m_0\}$  can be identified with  $\{z \in \mathbb{C} : |z| \leq 1 - 2/m_0\}$ . Finally, since  $\tilde{\omega}_{m_0}(t)$  is always a coordinate of  $\omega(t)$ ,  $\tilde{\omega}_{m_0}(t) \in \bar{W}$  for all  $t \in [0, t_{m_0}]$ .

By the same process, we inductively find for each  $m > m_0$  a path  $\tilde{\omega}_m : [t_{m-1}, 1] \rightarrow M_m$  such that  $\tilde{\omega}_m(t_{m-1}) = \tilde{\omega}_{m-1}(t_{m-1})$ . Let  $t_m$  be the smallest  $t \geq t_{m-1}$  such that

$$|\tilde{\omega}_m(t)| \geq 1 - \frac{2}{m},$$

and obtain a path

$$\tilde{\omega}_m : [t_{m-1}, t_m] \rightarrow \bar{W}$$

which we append to  $\tilde{\omega}_{m-1}$  to obtain

$$\tilde{\omega}_m : [0, t_m] \rightarrow \bar{W}$$

such that  $\tilde{\omega}_m(t)$  is always a coordinate of  $\omega(t)$ .

Let  $t_\infty = \sup_m t_m$ . Piecing together the  $\tilde{\omega}_m$ 's gives a continuous map

$$\tilde{\omega} : [0, t_\infty) \rightarrow \bar{W}$$

such that  $\tilde{\omega}(t)$  is a coordinate of  $\omega(t)$  for all  $t \in [0, t_\infty)$ . The set of limit points of  $|\tilde{\omega}(t)|$  as  $t \rightarrow t_\infty$  is a closed interval  $I$ . Let  $\omega(t_\infty) = \{z_1, z_2, z_3, \dots\}$ . If  $r \in [0, 1)$  is distinct from  $|z_1|, |z_2|, |z_3|, \dots$  then  $\sup_i |r - |z_i|| > 0$  (since  $\{z_i\} \rightarrow P$ ) and by continuity of  $\omega$ ,  $r$  also differs by some  $\varepsilon$  from all coordinates of  $\omega(t)$  for  $t$  in a neighborhood of  $t_\infty$ , so  $r$  cannot be a limit point of  $|\tilde{\omega}(t)|$  as  $t \rightarrow t_\infty$ . Thus  $I \subseteq \{1, |z_1|, |z_2|, \dots\}$  but  $|z_i| \rightarrow 1$  so  $I$  must be a single point. Since  $|\tilde{\omega}(t_m)| = 1 - 2/m$  for  $m \geq m_0$ ,  $\lim_{t \rightarrow t_\infty} |\tilde{\omega}(t)| = 1$ .

*Case 1:* 1 is the only limit point of  $\tilde{\omega}(t)$  as  $t \rightarrow t_\infty$ . Then  $\tilde{\omega}$  extends to a path  $[0, t_\infty] \rightarrow \bar{W}$  from  $z_0$  to 1.

*Case 2:* There is a limit point  $\theta \neq 1, |\theta| = 1$ , of  $\tilde{\omega}(t)$  as  $t \rightarrow t_\infty$ . By Theorem 3.1, there is an open disc centered at  $\theta$  contained in  $\bar{W}$ . For some  $t < t_\infty$ ,  $\tilde{\omega}(t)$  is in this disc, so we can replace the tail end of  $\tilde{\omega}$  on  $[t, t_\infty)$  by a straight line from  $\tilde{\omega}(t)$  to  $\theta$  in  $\bar{W}$ .

In either case we can connect  $z_0$  to a point on the unit circle via a path in  $\bar{W}$ . The same is true if  $z_0 \in \bar{W}, |z_0| > 1$ , since  $\bar{W}$  is closed under  $z \mapsto 1/\bar{z}$ . Since  $\bar{W}$  contains the unit circle, this proves that  $\bar{W}$  is path connected.  $\square$