1. Background in geometric group theory

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AUTOMATIC GROUPS: A GUIDED TOUR

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Geometric group theory has seen a flurry of activity in the last ten years due to new geometric ideas of Gromov, Cannon, Thurston and others. This resurgence has given birth to a truly new field of mathematics: the theory of automatic groups. This fast-growing field lies at the juncture of geometry, topology, combinatorial group theory and algorithms; many of its ideas and themes have their roots in mathematics from throughout this century. As this subject is nearing the end of its infancy it seems an appropriate time for a quick survey. The writing of this paper grew out of conversations with the curious, talks given at Cornell and Princeton, and an attempt to set things straight in my own mind. I'll try to give the reader a taste of some of the main ideas, techniques and applications of the theory. For a detailed, comprehensive introduction to automatic groups, the reader may consult the upcoming book *Word Processing and Group Theory* by Epstein, Cannon, Holt, Levy, Paterson and Thurston ([E *et al.*]).

1. BACKGROUND IN GEOMETRIC GROUP THEORY

The theory of automatic groups is based on the study of groups from a geometric viewpoint. In order to put things in their proper perspective we'll first have to review a small bit of background material. To a group G with finite generating set S, one associates the *Cayley graph* $\Gamma_S(G)$ of (G, S), which is a directed graph whose vertex set consists of elements of G, with a directed edge labelled s going from g to $g \cdot s$ for each $g \in G, s \in S$. As a matter of convenience, for elements $s \in S$ which have order two, we draw only one (undirected) edge labelled s between g and $g \cdot s$, as opposed to drawing one from g to $g \cdot s$ and another from $g \cdot s$ to g. The Cayley graph can be made

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into a metric space by assigning each edge length 1, and by defining the distance between two points to be the length of the shortest path between them. A word representing $g \in G$ can be thought of as a path in $\Gamma_S(G)$ from 1 to g; a geodesic in $\Gamma_S(G)$ from g to h is the same as a word of minimal length representing $g^{-1}h$. Examples of Cayley graphs of some familiar groups are given in figure 1. It is possible to construct and study geometric objects such as geodesics and triangles in the metric space $\Gamma_S(G)$. One theme of geometric group theory is that there is an interplay between geometric properties of the Cayley graph and group theoretic properties of G.

For general finitely presented groups (i.e., groups having a presentation with finitely many generators and finitely many relators), not much can be said. Recall that if G is a finitely presented group with generating set S, the "word problem" for G consists of giving an algorithm which takes as input any two elements of the free group on S (so-called 'words'), and as output tells whether or not the two words represent the same element of G; equivalently, there is an algorithm which tells whether or not an input word represents the identity element of G. In one of the great mathematical achievements of the 1950's, Novikov and Boone found a finitely presented group for which the word problem is not solvable. We shouldn't give up so easily, though, since we are mostly interested in studying groups that arise in geometry and topology, and it is in these situations where we can hope to use the structure of the spaces to tell us more about the groups.

If G arises naturally from some geometric situation, for example if G is the fundamental group of some compact hyperbolic manifold M, then the geometry of the Cayley graph $\Gamma_S(G)$ is in some sense a combinatorial model of the geometry of M. Geodesics, spaces at infinity, and global manifestations of curvature are all captured by the metric space $\Gamma_S(M)$. In fact, if G is the fundamental group of a compact Riemannian manifold M, then $\Gamma_S(G)$ is quasi-isometric (i.e., isometric up to constant factors) to the universal cover $\tilde{M}$ of M. The study of the geometry of the Cayley graph and its group theoretic implications is part of the field of geometric group theory. For an inspiring introduction to some of this material, see [Ca2]. As Cannon notes in his paper, one of the central philosophies of geometric group theory is that geometric models of groups give rise to computational schemes for dealing with those groups. It is this idea that we shall explore.