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LINK SIGNATURE, GOERITZ MATRICES AND POLYNOMIAL INVARIANTS

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ABSTRACT

Lickorish and Millett introduced the notion of skein equivalence of oriented links in [15]. In the first part of this paper I show that the link signature function $\sigma_L : S^1 \to Z$ of [22], [12], etc. is a skein invariant for links with non-zero Alexander polynomial. In the second part I show that a renormalised form of Kauffman's polynomial invariant $F_L(a, z)$, well-defined on non-oriented links is calculable from the Goeritz matrix.

1. P-SKEINS AND SIGNATURE

In this section I present two notions of "skein equivalence" for links; "broad" skein equivalence and "narrow" skein equivalence. Narrow skein equivalence is a stronger relation (i.e. has smaller equivalence classes), but it is not clear whether it is strictly stronger. I show that the link signature function $\sigma_L : S^1 \to Z$ is a broad skein invariant for all links with non-zero Alexander polynomial. It is not known whether this result extends to links with zero Alexander polynomial, but it seems unlikely that it should.

1.1. PRELIMINARIES

I briefly recap on some standard material. See [19], [2] or [5] for further details. Let L be an oriented link. Then it is always possible to find an oriented surface F embedded in S^3 in such a way that $\partial F = L$, with the appropriate orientation. Such a surface is called a *Seifert surface* for the link L. Now let $c_1, \ldots, c_n$ be closed curves lying in S whose homology classes generate $H_1(S)$, and let $c_1^+, \ldots, c_n^+$ be the results of pushing these curves slightly away from S in the positive direction in a collar neighbourhood of the surface. The matrix $V = (v_{ij})$, where $v_{ij} = lk(a_i, a_j^+)$, is called a *Seifert matrix* for the link L. Of course such a matrix is not well-defined, but it is true that any two Seifert matrices for a link L are *S-equivalent*; that is, can be transformed into each other by a finite sequence of the moves

(1) $$V \to PVP',$$

where P is an integral unimodular matrix, and

(2) $$V \to \begin{pmatrix} V & & \\ & 0 & 1 \\ & 0 & 0 \end{pmatrix}.$$

Up to sign and multiplication by a power of t, the Alexander polynomial can be obtained by

(3) $$\Delta_L(t) \doteq \det(tV - V').$$

Finally, the *determinant* of L is given by

(4) $$\det(L) = i^n \det(V + V')$$

where V is an $n \times n$ square matrix, and the classical *signature* of the link L is defined by

(5) $$\sigma_L = \sigma(V + V')$$

and this turns out to be a well-defined link invariant.

Tristram [22], Levine [12] and others have developed the following generalisation of the classical signature of a link: Let V be a Seifert matrix for a link L, and ω a complex number of modulus 1. Instead of considering the symmetric matrix $V + V'$, we can deal with the Hermitian matrix $H(\omega) = (1-\bar{\omega})V + (1-\omega)V'$. The fact that $|\omega| = 1$ enables us easily to show that the signature $\sigma(H(\omega))$ is unchanged by the moves (1) and (2) and so takes the same values for S-equivalent matrices. It hence provides a function $\sigma_L: S^1 \to Z$ which may be regarded as a link invariant. Further, $H(\omega) = (\omega-1)(\bar{\omega}V - V')$ so $\sigma_L(\omega)$ is continuous away from roots of the Alexander polynomial. Clearly $\sigma_L(-1)$ is twice the classical signature of a link.

By a *skein triplet* (or *oriented skein triplet*) of links I shall mean a triplet (L_+, L_-, L_0) of oriented links which are identical outside some ball $B \subset S^3$ and inside it are as shown in Figure 1.

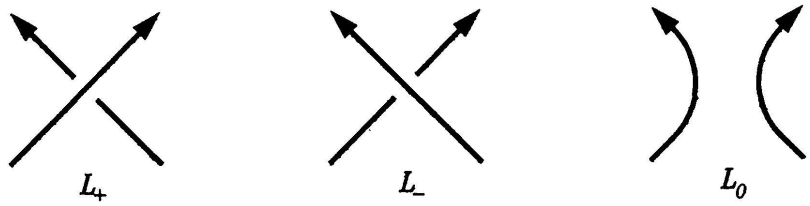


FIGURE 1

There are two equivalence relations with which we may be interested: Let R_b be the equivalence relation on the set of links generated by:

$$\text{(6)}\quad \begin{array}{c}\text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0)\\ \text{are skein triples then}\end{array} \left\{\begin{array}{l} L_+R_bL'_+,\, L_-R_bL'_- \Rightarrow L_0R_bL'_0 \\ L_+R_bL'_+,\, L_0R_bL'_0 \Rightarrow L_-R_bL'_- \\ L_-R_bL'_-,\, L_0R_bL'_0 \Rightarrow L_+R_bL'_+ . \end{array}\right.$$

I shall call this equivalence relation *broad oriented skein equivalence.* The other relation on the set of oriented links, *narrow oriented skein equivalence*, is the equivalence relation R_n generated by:

$$\text{(7)}\quad \begin{array}{c}\text{if } (L_+, L_-, L_0) \text{ and } (L'_+, L'_-, L'_0)\\ \text{are skein triples then}\end{array} \left\{\begin{array}{l} L_+R_nL'_+,\, L_0R_nL'_0 \Rightarrow L_-R_nL'_- \\ L_-R_nL'_-,\, L_0R_nL'_0 \Rightarrow L_+R_nL'_+ \end{array}\right. .$$

It is obvious that R_b is a weaker equivalence relation than R_n (i.e. the equivalence classes are larger), but it is not clear (and I do not know) whether it is strictly weaker. By the *broad* or *narrow oriented skein of links* I refer to the set of equivalence classes of oriented links under the relation R_b or R_n (Note that in most of the literature, R_n is referred to as "skein equivalence"; R_b is not referred to at all). The polynomial invariant $P_L(l, m)$ of [15], [3] etc. may be regarded as the most general linear broad skein invariant (see [15], [16]). The fact that the value of $P_L(l, m)$ specified on the unknot U is sufficient to define its value on any link may be taken as saying that the broad oriented skein is *generated* by U. The corresponding statement for the narrow oriented skein is that specifying the values of $P_L(l, m)$ on all unlinks is sufficient to define its values on all links — the set of unlinks generates the narrow oriented skein.

1.2. SIGNATURE AND ORIENTED SKEINS

I now show that the signature function $\sigma_L(\xi)$ of any link with non-zero Alexander polynomial is a broad oriented skein invariant (It is already known that the signature $\sigma = \frac{1}{2}\sigma_L(-1)$ is a narrow skein oriented invariant for knots — which are, of course a proper subset of the set of links with non-zero Alexander polynomial — see e.g. [15]). Before proceeding with this, however, I introduce some convenient notation. Given links L and L', set

$$\rho_0(L, L') = \begin{cases} \infty & \text{if } L \neq L' \\ 0 & \text{if } L = L' \end{cases}. \tag{8}$$

Now for $n > 0$ set

$$\rho_{n+1}(L, L') = \min \begin{cases} \min\limits_{\substack{\text{skein triples} \\ (L, L_1, L_2) \\ (L', L'_1, L'_2)}} (1 + \max\{\rho_n(L_1, L'_1), \rho_n(L_2, L'_2)\}) , \\ \min\limits_{\substack{\text{skein triples} \\ (L_1, L, L_2) \\ (L'_1, L', L'_2)}} (1 + \max\{\rho_n(L_1, L'_1), \rho_n(L_2, L'_2)\}), \\ \min\limits_{\substack{\text{skein triples} \\ (L_1, L_2, L) \\ (L'_1, L'_2, L)}} (1 + \max\{\rho_n(L_1, L'_1), \rho_n(L_2, L'_2)\}) , \\ \min\limits_{\text{Links } L''} (\rho_n(L, L'') + \rho_n(L', L'')) . \end{cases} \tag{9}$$

Finally, define

$$\rho(L, L') = \lim_{n \to \infty} \rho_n(L, L') \tag{10}$$

to be the *broad skein distance* from L to L'. It is easy to see that ρ is a metric on the set of links, and that $\rho(L, L') < \infty$ if and only if L and L' are broadly skein equivalent. Intuitively, ρ measures the number of skein triples in a minimal chain of simple skein equivalences needed to establish skein equivalence. It is useful because it provides a grip on broad skein equivalence for use in inductive proofs.

By modifying equation (9) (i.e. by removing the third line inside the outer minimum) a similar metric can easily be defined for narrow skein equivalence, but I shall not need it here.

Note that since skein-equivalent links have the same $P_L(l, m)$ and hence $\Delta_L(t)$ and determinant, it makes sense to speak of the determinant of a skein equivalence class.

THEOREM 1. *The signature is well-defined on broad oriented skein equivalence classes with non-zero determinant.*

Proof. The proof proceeds by induction on the broad oriented skein distance ρ between two links. If $\rho(L, L') = 0$ then $L = L'$ and trivially $\sigma(L) = \sigma(L')$. For the inductive hypothesis suppose that $\rho(L, L') < n$ implies that $\sigma(L) = \sigma(L')$ whenever $\det(L) = \det(L') \neq 0$. I will prove that this is enough to show that the same is true whenever $\rho(L, L') = n$ and $\det(L) = \det(L') \neq 0$. This will suffice to prove the theorem.

Now if $\rho(L, L') = n$ then from the construction of ρ we can see that $\rho_n(L, L') = n$ and $\rho_{n-1}(L, L') = \infty$. One of the following cases must occur, corresponding to the four lines in the outer minimum of equation 9:

1) There exist skein triplets (L, L_1, L_2) and (L', L'_1, L'_2) such that $\rho(L_1, L'_1) \leqslant n - 1$ and $\rho(L_2, L'_2) \leqslant n - 1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$, provided $\det(L_1) \neq 0$ and $\det(L_2) \neq 0$ respectively. Let L, L_1, L_2 be given Seifert surfaces M_+, M_-, M_0 which are identical away from the crossing at which the links differ and near it are as shown in Figure 2.

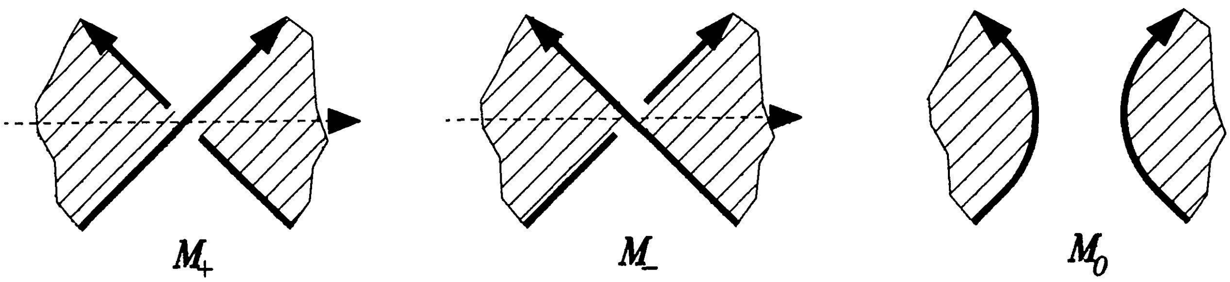


FIGURE 2

Choose a set of generators for $H_1(M_0; Z)$ and extend to sets of generators for the first homologies of M_+ and M_- by including loops which intersect the additional crossing once in the link projection as shown, and which are identical away from this crossing. Then using these to obtain Seifert matrices V_+, V_-, V_0 for the three links, we find that $V_+ + V'_+$, $V_- + V'_-$, $V_0 + V'_0$ are of the form

$$\begin{pmatrix} 2r & \rho \\ \rho' & S_0 \end{pmatrix}, \begin{pmatrix} 2r + 2 & \rho \\ \rho' & S_0 \end{pmatrix}, (S_0). \tag{11}$$

Now if $\det(L_2) \neq 0$, transformations of the form $Q \to PQP'$ (P nonsingular and with rational entries) suffice to put these matrices in the form

$$\begin{pmatrix} m & 0 \\ 0 & S_0 \end{pmatrix}, \begin{pmatrix} m - 1 & 0 \\ 0 & S_0 \end{pmatrix}, (S_0), \tag{12}$$

where $\det(S_0) \neq 0$. Then

$$\begin{aligned}
\sigma(L) &= \sigma(V_+ + V'_+) \\
&= \sigma(S_0) + \operatorname{sgn}(m) \\
&= \sigma(L_2) + \operatorname{sgn}\left(\frac{\det(L)}{i\det(L_2)}\right) \\
&= \sigma(L'_2) + \operatorname{sgn}\left(\frac{\det(L')}{i\det(L'_2)}\right) \\
&= \sigma(L')
\end{aligned}$$

using the inductive hypothesis for the equality of the third and fourth lines. If, however, $\det(L_2) = 0$, then $\det(L) \neq 0 \Rightarrow \det(L_1) \neq 0$ and the three matrices can be put in the form

$$(14) \qquad \begin{pmatrix} m & 0 & \varepsilon \\ 0 & A & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} m-1 & 0 & \varepsilon \\ 0 & A & 0 \\ \varepsilon & 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix},$$

with $\det(A) \neq 0$ and $\varepsilon = \pm 1$, whence it is easy to check that

$$(15) \qquad \sigma(L) = \sigma(A) = \sigma(L_1) = \sigma(L'_1) = \sigma(L').$$

2) There exist skein triplets (L_1, L, L_2) and (L'_1, L', L'_2) such that $\rho(L_1, L'_1) \leqslant n-1$ and $\rho(L_2, L'_2) \leqslant n-1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$. We may prove by precisely the same arguments as in Case 1) that $\sigma(L) = \sigma(L')$.

3) There exist skein triplets (L_1, L_2, L) and (L'_1, L'_2, L') such that $\rho(L_1, L'_1) \leqslant n-1$ and $\rho(L_2, L'_2) \leqslant n-1$. Then $\sigma(L_1) = \sigma(L'_1)$ and $\sigma(L_2) = \sigma(L'_2)$, provided $\det(L_1) \neq 0$ and $\det(L_2) \neq 0$ respectively. In this case $L = L_0$ in the skein triplet under construction. As above, we may choose Seifert matrices V_+, V_-, V_0 for L_1, L_2, L such that (after transformations of the form $Q \to PQP'$, P non-singular and with rational entries) $V_+ + V'_+, V_- + V'_-, V_0 + V'_0$ take the forms

$$(16) \qquad \begin{pmatrix} m & 0 \\ 0 & S_0 \end{pmatrix}, \begin{pmatrix} m-1 & 0 \\ 0 & S_0 \end{pmatrix}, (S_0),$$

where $\det(S_0) \neq 0$. Then at least one of L_1, L_2 has non-zero determinant. Without loss of generality, suppose $\det(L_1) \neq 0$. Then

$$
\begin{aligned}
\sigma(L) &= \sigma(S_0)\\
&= \sigma(L_1) - \operatorname{sgn}(m)\\
&= \sigma(L_1) - \operatorname{sgn}\left(\frac{\det(L_1)}{i\det(L)}\right) \qquad (17)\\
&= \sigma(L'_1) - \operatorname{sgn}\left(\frac{\det(L'_1)}{i\det(L')}\right)\\
&= \sigma(L')
\end{aligned}
$$

using the inductive hypothesis to establish the equality of the third and fourth lines.

4) There exists a link L'' such that $\rho(L, L'') < n$ and $\rho(L'', L') < n$. Then $\sigma_{L'} = \sigma_{L''} = \sigma_L$.

This concludes the proof of Theorem (1). Note the use made of the fact that it is not possible for exactly one member of a skein triplet to have non-zero determinant. Note also that although the transformations $Q \to PQP'$ (P nonsingular with rational entries) may alter the determinant, they do not change its sign, which is all that the proof requires. □

Unfortunately it is not clear how to deal with the cases in which $\det(L) = 0$, because it is conceivable that one or more of the skein triplets in the chain establishing skein equivalence of two links could have all three determinants equal to zero, in which case the methods of the above proof would not be applicable. It becomes clearer what is going on and that these exceptions are not just artifacts of a poor proof when the more general situation of the signature function $\sigma_L\colon S^1 \to Z$ is considered.

Now since the Alexander polynomial $\Delta_L(t)$ can be obtained from $P_L(l, m)$ it is clearly a broad oriented skein invariant and it makes sense to state

THEOREM 2. *Broadly skein-equivalent oriented links have the same signature function $\sigma_L(\omega)$ for all ω other than roots of the Alexander polynomial.*

Proof. The proof of Theorem 4 goes through virtually unmodified ($\Delta_L(\omega)$ takes the place of the determinant, and the obvious changes are made to accommodate the fact that we are dealing with Hermitian matrices instead of symmetric ones). □

Now if we adopt the usual convention (see [5]) that the value of $\sigma_L(\omega)$ at a root of the Alexander polynomial is defined to be the mean of its two "adjacent" values

$$\lim_{\varepsilon \to 0+} \sigma_L(\omega e^{i\varepsilon}) \quad \text{and} \quad \lim_{\varepsilon \to 0-} \sigma_L(\omega e^{i\varepsilon}), \tag{18}$$

the fact that both of these values are well-defined broad oriented skein invariants completes the proof that

COROLLARY 3. *The signature function* $\sigma_L : S^1 \to Z$ *is a broad oriented skein invariant for all links with non-zero Alexander polynomials.* □

This is an intriguing result, especially in view of the fact that $\sigma_L(\omega)$ is known to be a concordance invariant. It is natural to ask what relations there may be between skein theory and concordance theory. Another obvious question is that of what happens when the Alexander polynomial Δ_L is identically zero. In these circumstances the first Alexander ideal of the link collapses and the signature function can be thought of as extracting information about higher Alexander ideals. Kanenobu ([8] and [9]) has shown that there exist infinitely many links with identical P-polynomials but distinct second Alexander ideals, so there is no obvious reason to suppose that this information should be skein invariant. However, I know of no counter-examples to the conjecture that $\sigma_L(\omega)$ is a broad oriented skein invariant for all links.

2. GOERITZ MATRICES AND THE F-POLYNOMIAL

In this section I explore the relationships between the graph of a link, its Goeritz matrix and Kauffman's polynomial invariant $F_L(a, z)$. In particular I show that the $F(a, z)$, is essentially calculable from the Goeritz matrix of a knot. This result makes use of facts about planar graphs discovered by Whitney over 50 years ago.

2.1. THE GOERITZ MATRIX AND GRAPH OF A LINK

Kauffman [10] has defined a polynomial invariant $F_L(a, z)$ of oriented links as follows:

Recall the definition of the three *Reidemeister moves,* see Figure 3.

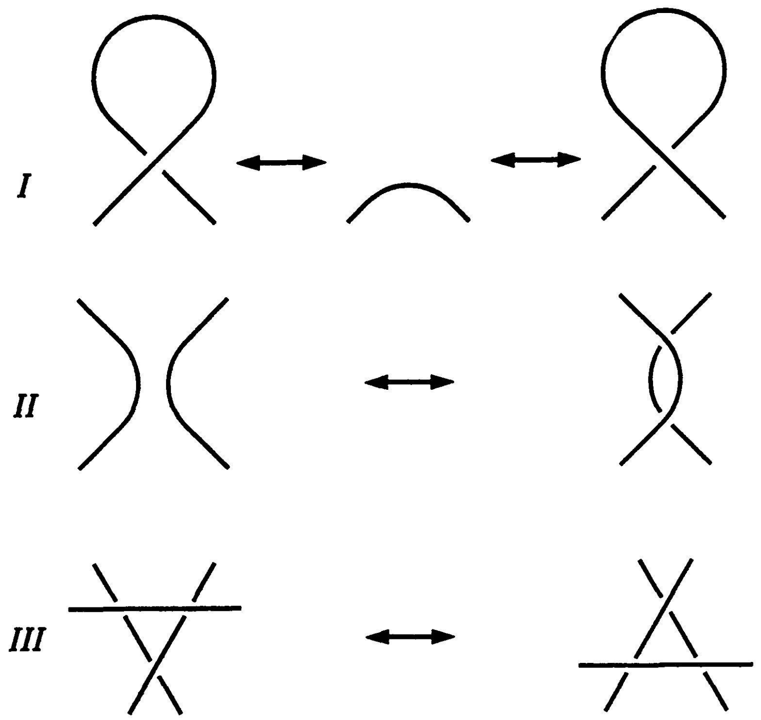


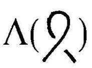
FIGURE 3

Two link diagrams represent the same link if and only if one can be transformed into the other by a finite sequence of these moves (see [18]). We define a polynomial invariant $\Lambda \in Z[a^{\pm 1}, z^{\pm 1}]$ of unoriented link diagrams by the four axioms:

i) Λ (unknot) $= 1$.

ii) Λ is invariant under Reidemeister moves II and III.

iii) The effect of Reidemeister move I on Λ is to multiply by a or a^{-1}:

(19) $\Lambda($$) = a^{-1}\Lambda($$)$, $\quad\Lambda($$) = a\Lambda($$)$.

iv) If four link diagrams L_+, L_-, L_0 and L_∞ are identical except in a ball B where they are as shown in figure 4 then

$$\Lambda(L_+) + \Lambda(L_-) = z(\Lambda(L_0)+\Lambda(L_\infty)) \tag{20}$$

Axioms i)-iv) are sufficient to define Λ for all link diagrams. Now given an oriented diagram we can temporarily forget its orientation and calculate its Λ-polynomial. Let w be the *writhe* of the diagram (that is, the number of positive crossings less the number of negative crossings). Then

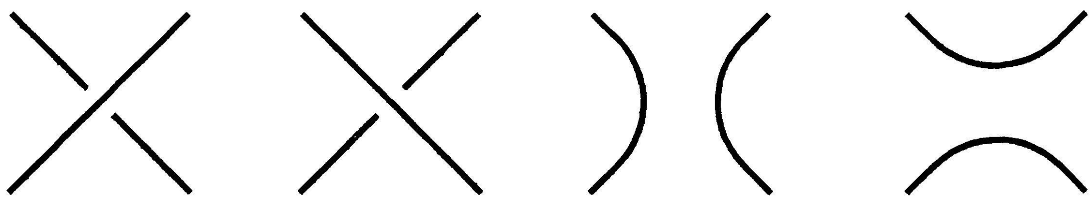
$$F(a, z) = \Lambda(a, z)\cdot a^{-w} \tag{21}$$

is a link invariant, the *Kauffman polynomial* (see [10]). Note that in order to define F_L we need the writhe, which is orientation-dependant, so F_L is an invariant of oriented links. However, for knots (1-component links), reversing the orientation leaves the sign of any given crossing, and hence the writhe, unchanged, so for knots F_L may be regarded as an unoriented invariant. If $L = \bigcup_{i=1}^{n} L_i$ is an arbitrary oriented link with components L_i, then $F_L(a, z) \cdot a^{\lambda/2}$, where $\lambda = \sum_{i \neq j} lk(L_i, L_j)$ is the *total linking number of* L is unchanged by reversal of orientations of components, and so should be regarded as an unoriented link invariant (This observation has also been made by Turaev in [23]). Like $P_L(l, m)$, $F_L(a, z)$ behaves nicely with respect to disjoint and connected sums of links:

(22) $$F_{L_1\ L_2}(a, z) = F_{L_1}(a, z) \cdot F_{L_2}(a, z)$$

(23) $$F_{L_1 \cup L_2}(a, z) = z^{-1}(a + a^{-1} - 1) \cdot F_{L_1}(a, z) \cdot F_{L_2}(a, z)$$

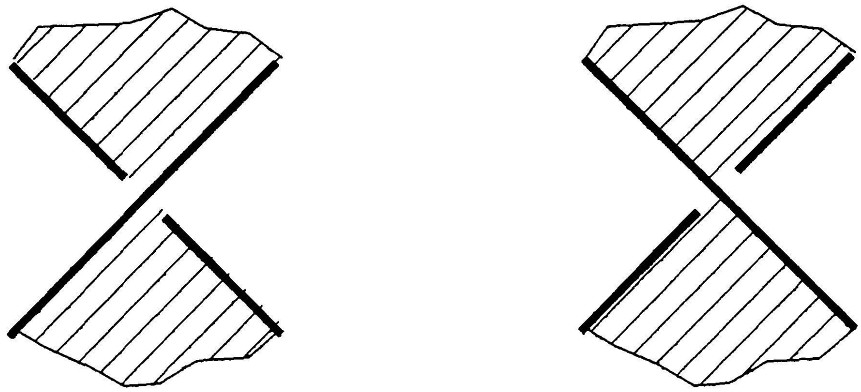
and is also invariant under mutation (see [1], [10]).



L_+ L_- L_0 L_∞

FIGURE 4

Now recall the definition of the *Goeritz matrix* of an unoriented connected link diagram $\mathscr{D}$ in the plane (see [6]). Such a diagram divides the plane into regions, which we proceed to colour black and white, chess board fashion, so that adjacent regions are distinct colours (It is not hard to see, using the Jordan curve theorem, that this can always be done). By convention we colour the infinite region white. Now label the black regions $R_1, R_2, \ldots, R_n$ say. At each crossing in the diagram a region R_i meets a region R_j, not necessarily distinct. This crossing takes one of two forms, illustrated in Figure 5, and we allocate signs $\xi = \pm 1$ to the crossings accordingly (The value ξ, which is defined only in the presence of a chess-board colouring of a link diagram, should not be confused with the sign of a crossing as defined in section 1. Unfortunately, the word "sign" is now well-established for each of these values in the literature.)



$\xi = +1$ $\xi = -1$

FIGURE 5

Now construct an $(n \times n)$ matrix A as follows: for $i \neq j$, let $a_{ij} = \sum \xi(c)$, where the sum is over all crossings c at which R_i meets R_j in the diagram. The diagonal elements are given by $a_{ii} = -\sum_{i \neq j} a_{ij}$ so that the row and column sums are all zero. A Goeritz matrix G for the link is then obtained by discarding the first row and column of A. Clearly G is not an invariant of the link, or even of the link diagram (any other row or column could have been discarded instead of the first one, for example). It is, however, a relation matrix for $H_1(D_L)$, where D_L is the two-fold branched cover of the link complement, and certain functions of it are true link invariants. For instance, the absolute value of its determinant is the absolute value of the determinant of the link. Further, G^{-1} is a matrix of the linking form on $H_1(D_L)$. I shall show later in this section that, up to the writhe and total linking number, Kauffman's two-variable polynomial $F_L(a, z)$ is a function of G. This raises the (unanswered) question of precisely what $F_L(a, z)$ has to do with, for example, this linking form.

The *graph* of a unoriented link diagram is constructed in a similar way. Take a vertex v_i in each black region R_i of the chess board coloured link diagram. Now for each crossing c at which R_i meets R_j, add an edge joining the corresponding vertices v_i, v_j. This edge is labelled with the sign $\xi(c)$ of the crossing. This construction provides us with a (signed) planar graph with a particular planar embedding. Conversely, given a planar embedding of a signed graph, one can construct a corresponding link diagram by placing a crossing of the appropriate sign in the middle of each edge and connecting these by arcs that run parallel to the edges of the graphs until they meet in neighbourhoods of the vertices. The graph is connected if and only if the link diagram is. See Figure 6 for an example and [2] for more details.

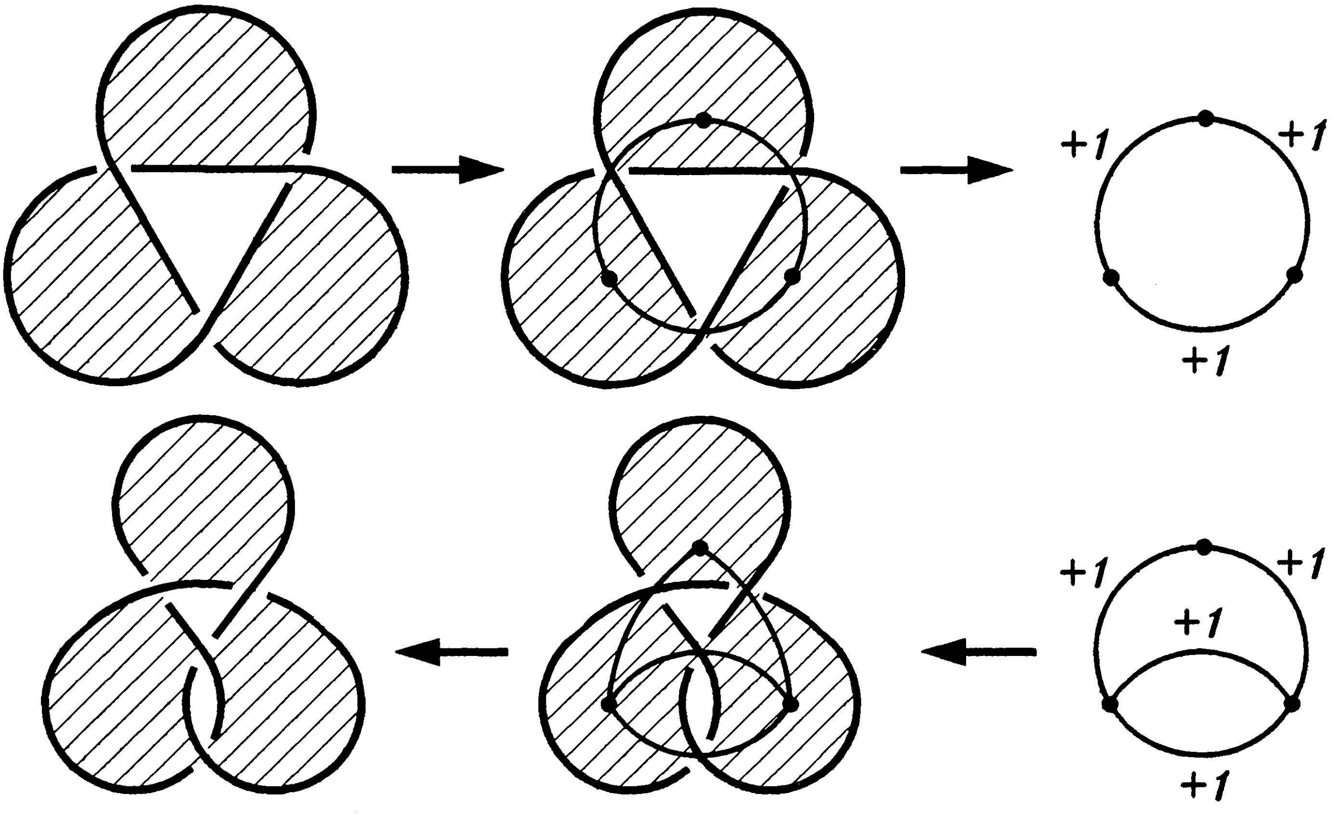


FIGURE 6

Notice that the graph of a connected diagram contains strictly more information than the Goeritz matrix, all information about the particular planar embedding and about loops in the graph being lost. Indeed, we can make use of this to construct diagrams of distinct links with identical Goeritz matrices, by picking graphs with more than one planar embedding. However, the variation that occurs here can be kept under tight control and I will make use of this fact later in this section.

2.2. KAUFFMAN'S POLYNOMIAL AND THE GOERITZ MATRIX

I now proceed to the main result of this section, linking the Goeritz matrix with Kauffman's F-polynomial invariant. Recall the observation made in section I that $\tilde{F}_L(a, z) = F_L(a, z) \cdot a^{\lambda/2}$ is invariant under change of orientation of components of L (where λ is defined to be the total linking number of L). Equivalently, we can define $\tilde{F}_L(a, z)$ by

$$\tilde{F}_L(a, z) = \Lambda_{\mathscr{D}}(a, z) \cdot a^{-w'} \tag{24}$$

where $\mathscr{D}$ is a diagram of the link L and w' is the *proper writhe* of $\mathscr{D}$, defined to be the algebraic sum of the signs of all crossings where a component of L meets itself. Note that the sign of such a crossing can be defined independently of any orientation on the link L (here I am speaking of crossing signs in the sense of section 1, not of the value ξ defined by a chess-board colouring of a diagram).

The following will be proved:

THEOREM 4. *The invariant* $\tilde{F}_L(a, z)$ *for a link* L *is a function of the Goeritz matrix of any diagram* $\mathscr{D}$ *of* L. □

Before proving Theorem 4, I must digress once again into graph theory. Recall that a graph is said to be *k-connected* if any $k - 1$ vertices (and their adjoining edges) may be removed without disconnecting the graph. The following result is due to Whitney ([27], [28]).

THEOREM 5. *Any planar embedding of a 3-connected graph is essentially unique.* □

The word "essentially" here means that we regard as equivalent any two embeddings which are ambient isotopic, any region of the graph's complement in the plane may be chosen to be the infinite region (this corresponds to a choice of region to contain the point at infinity in an embedding in the sphere) and the embedding may be reflected in some line in the plane. For more details, see [27], [28].

COROLLARY 6. *Let* P_1 *and* P_2 *be two planar embeddings of a connected graph* G. *Then there exists a finite sequence of the following moves which will transform* P_1 *into* P_2:

I. Ambient isotopy.

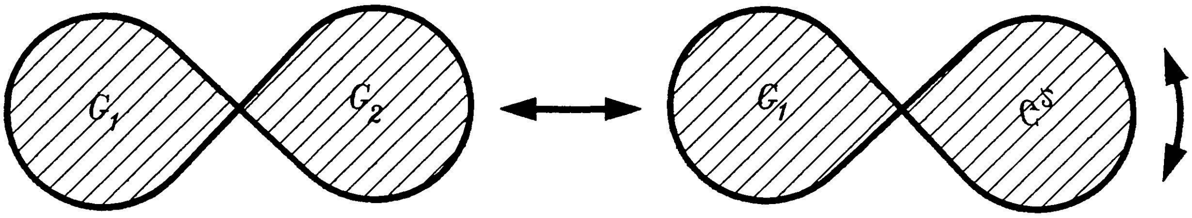
II. Reflection in a line.

III. The move illustrated in figure 7a).

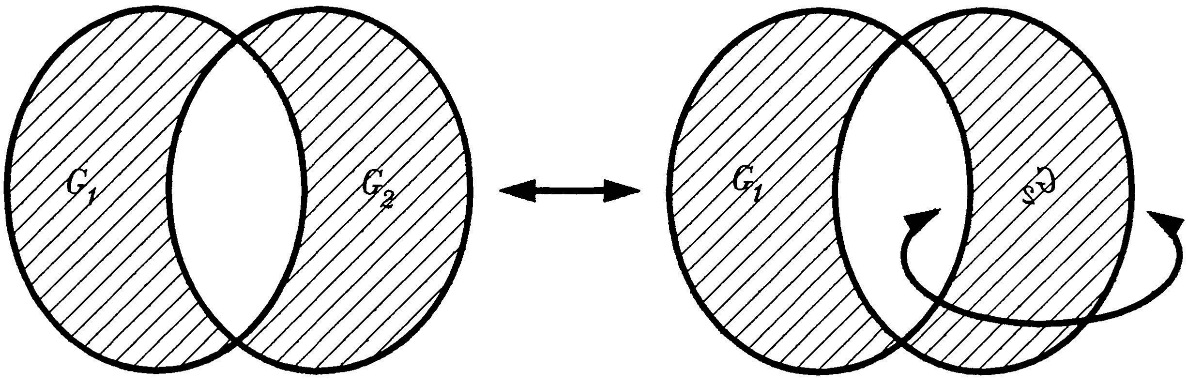
IV. The move illustrated in figure 7b).

V. The move illustrated in figure 7c).

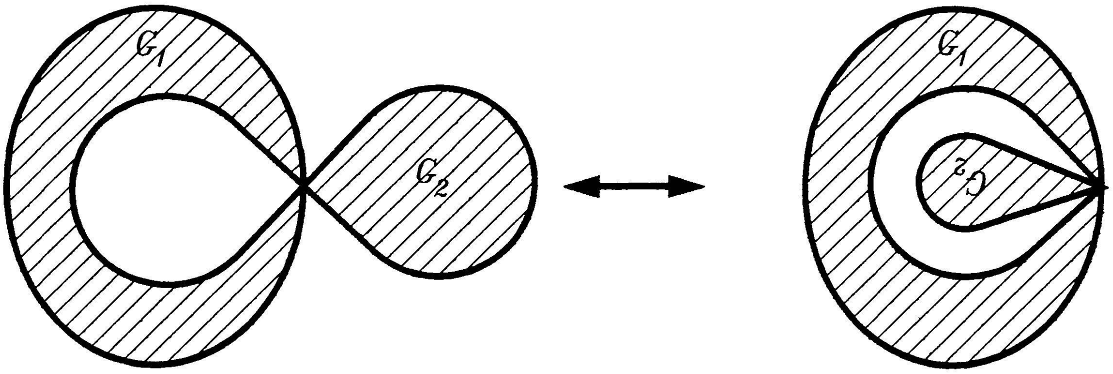
Proof. Proceed by induction on the number n of edges of G. Clearly the result is trivially true if $n = 0$. Now suppose it true far all connected graphs with $< n$ edges, and let G have n edges. If G is 3-connected then the result follows from Theorem 5. Otherwise there is a vertex or pair of vertices whose removal disconnects G. I consider these two cases separately.



a) Move III



b) Move IV



c) Move V

FIGURE 7

1) If there is a single vertex v whose deletion disconnects G, let G_1 and G_2 be the graphs obtained from the two components by adding to each a copy of v. Each of G_1, G_2 has fewer edges than G and so by the inductive hypothesis each satisfies the result. Now G is obtained from these two graphs by identifying the two copies of the vertex v and a planar embedding of G is specified by giving planar embeddings of G_1 and G_2 and specifying which planar region adjacent to the copy of v in each is occupied by the other. These different possibilities are all accounted for by moves III and V. Moves I, III, IV and V on G_1 and G_2 individually just correspond to the same moves on G, and move II on G_1 or G_2 corresponds to move III on G.

2) If there is a pair of vertices u and v whose removal separates G but no such single vertex, let G_1 and G_2 be the graphs obtained from the two components by adding to each copies of u and v. Each of G_1 and G_2 has fewer edges than G and so by the inductive hypothesis each satisfies the result. Now G is obtained from these two graphs by identifying the copies of u and v, and a planar embedding of G is specified by giving planar embeddings of G_1 and G_2 and specifying which planar region adjacent both to the copy of u and the copy of v in each is occupied by the other. These different possibilities are all accounted for by move IV. Moves I, III, IV and V on G_1 and G_2 correspond to the same moves on G, and move II on G_1 or G_2 corresponds to move IV on G. Hence the result is true of G and the induction proceeds. □

(In fact, move II is redundant since it follows from move IV, the subgraph on the left of the two chosen vertices in Figure 7b) consisting of a single edge joining those two vertices. Similarly, move III may be constructed from move IV, the subgraph to the left of the two chosen vertices in Figure 7b) being disconnected. I include these moves for clarity, however.)

Now consider the effects these moves on planar graphs have upon the corresponding link diagrams. Ambient isotopy of the graph merely corresponds to ambient isotopy of the link diagram. Reflection of the graph in some line corresponds to a reflection of the link diagram in that line followed by a reflection in the plane, the net effect of which is to rotate the link through 180 degrees about the line (see Figure (8)). Hence this does not change the link type corresponding to the graph's planar embedding.

Observe that if the signed graph G of a link diagram $\mathscr{D}$ has a cut-vertex v as in moves III and IV, then $\mathscr{D}$ is a connected sum. Figure 9 shows that move III corresponds to breaking up a connected sum and reconstituting it after reversing one of the summands. This may alter the link type of the connected sum (if at least one of the summands differs from its reverse), but does not affect the $\tilde{F}$-polynomial, since for any links L_1, L_2, we have

$$\tilde{F}_{L_1 \# L_2} = \tilde{F}_{L_1} \cdot \tilde{F}_{L_2}, \tag{25}$$

independent of the particular connected sum taken.

Similarly, move V corresponds to breaking up a connected sum and then reconstituting it, possibly summing together different components of the links. Again, this does not affect $\tilde{F}_L(a, z)$.

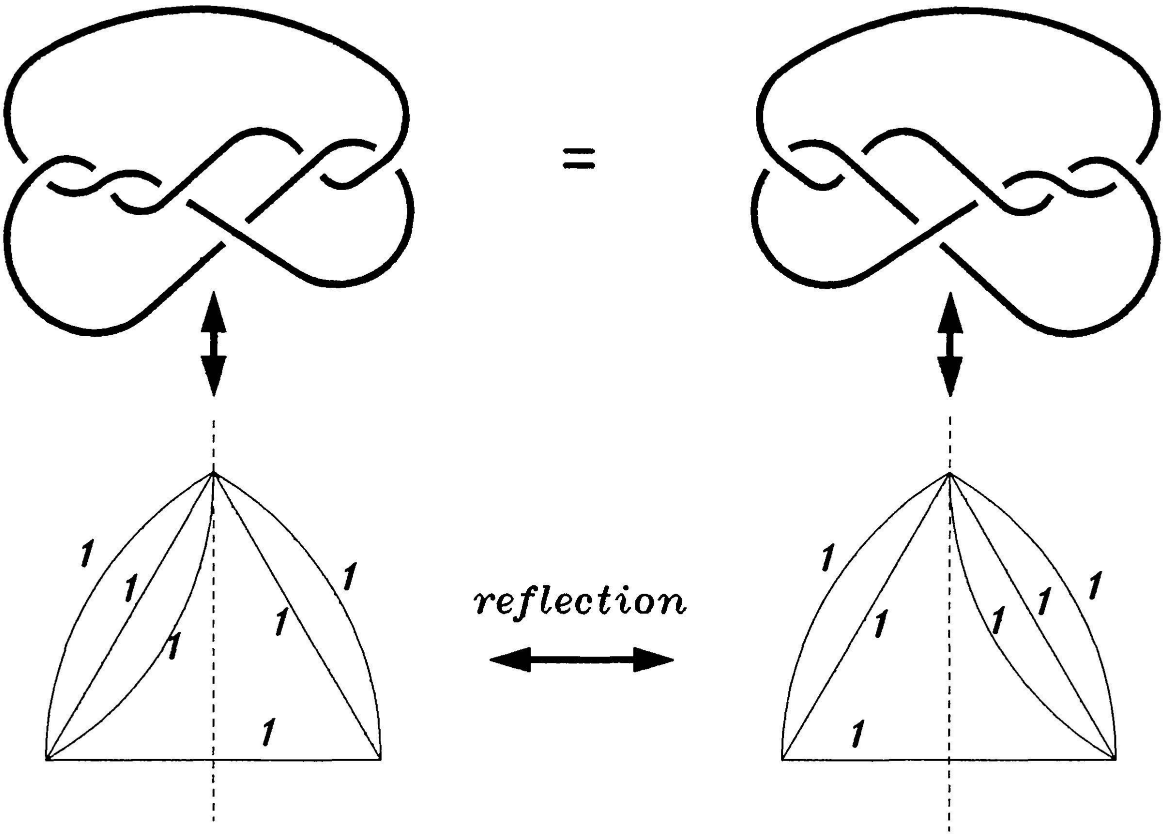


FIGURE 8

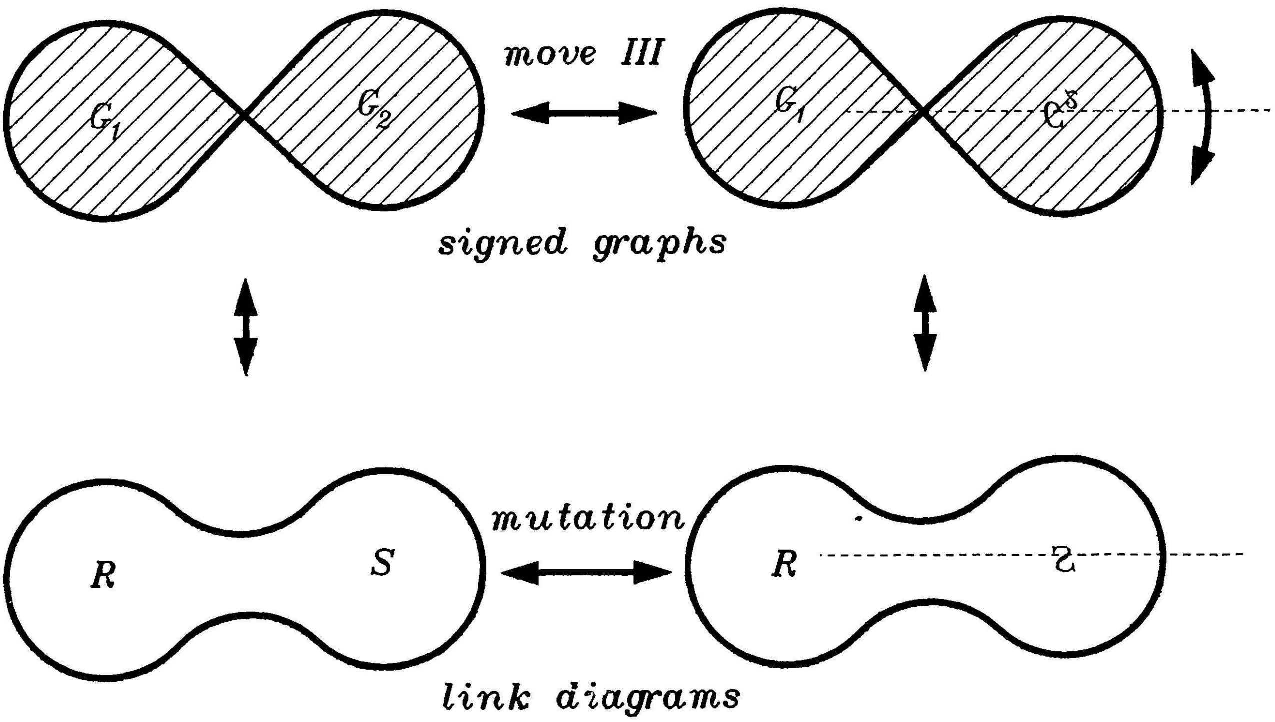


FIGURE 9

This leaves only move IV to be analysed. Figure 10 shows that this move corresponds to mutation of the underlying link. Once more, this leaves the $\tilde{F}$-polynomial unchanged.

The preceding discussion proves

THEOREM 7. *Given a link* L *with link diagram* $\mathcal{D}$, *the* $\tilde{F}$*-polynomial of* L *depends only on the isomorphism class of the signed graph corresponding to* $\mathcal{D}$, *and is independent of any particular planar embedding chosen.* □

In fact the same argument shows

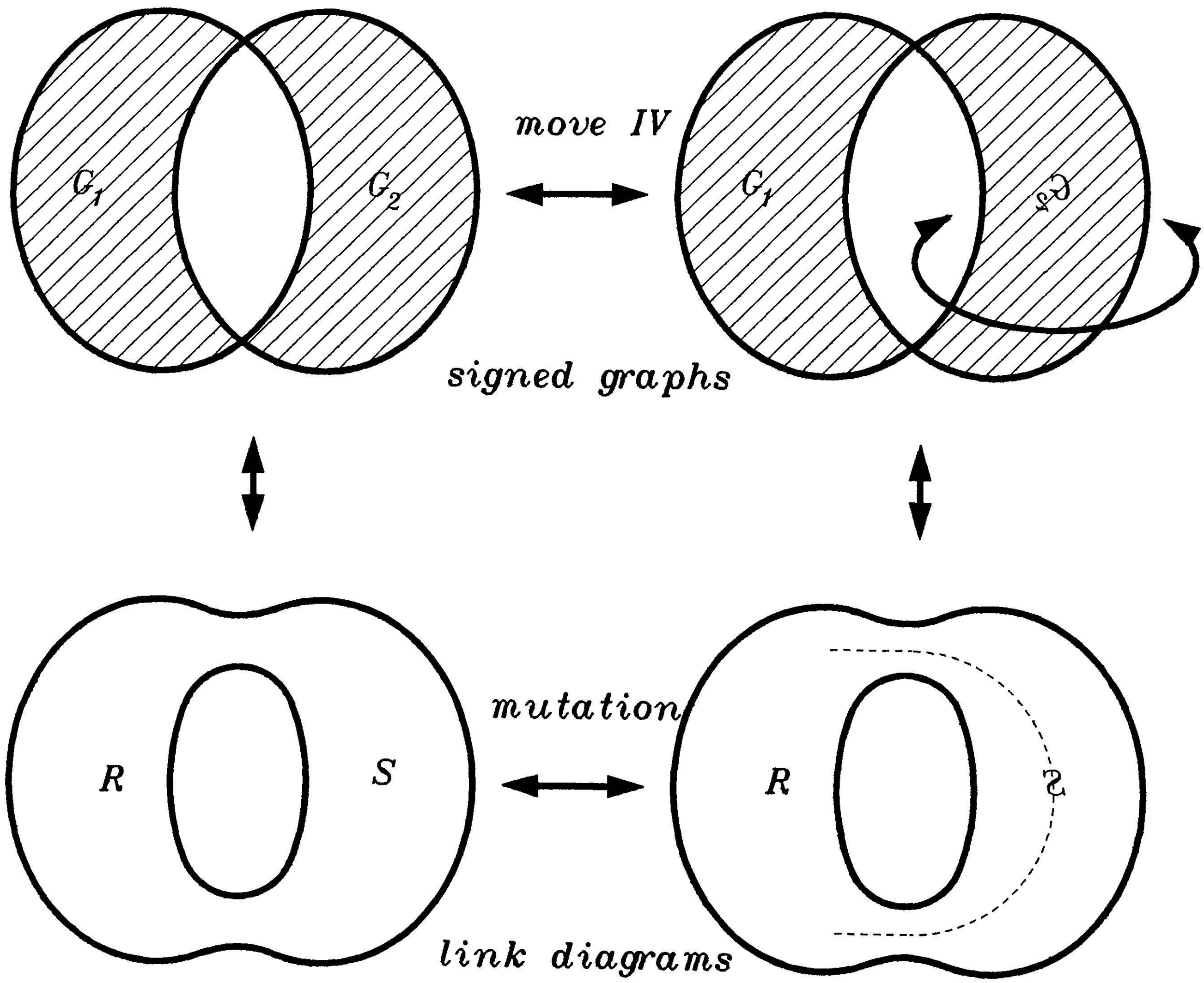


FIGURE 10

LEMMA 8. *Given a link diagram $\mathscr{D}$, the regular isotopy invariant $\Lambda_{\mathscr{D}}(a, z)$ depends only on the isomorphism class of the signed graph corresponding to $\mathscr{D}$, and is independent of any planar embedding information.* □

I can now proceed to the

Proof of Theorem 4. This follows from Theorem 7. The only information retained by the graph of a link diagram which is lost in passing to a Goeritz matrix is

(1) The number of edges of a given sign there are joining any particular pair of vertices. For each such pair the Goeritz matrix retains only the sum of the signs of these edges. But in terms of a chess-board colouring of the link diagram, this is to say that only the sum of the signs of crossings joining any two coloured regions R_i and R_j is retained. Suppose given a link diagram $\mathscr{D}$ with a chess-board colouring and two coloured regions R_i, R_j. Figure (11) shows that if R_i and R_j are connected by both a positive crossing and a negative crossing then by mutation of the link diagram these crossings can be made to cancel each other out.

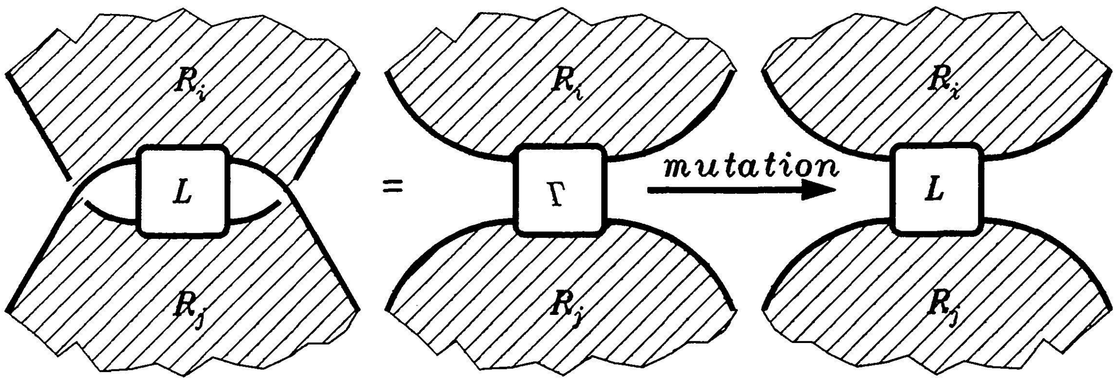


FIGURE 11

But mutation leaves $\tilde{F}_L(a, z)$ unaffected so it follows that only the sum of the signs of crossings joining each pair of coloured regions in $\mathscr{D}$ is relevant to calculation of $\tilde{F}_L$.

(2) The number of loops. However, loops in the graph correspond (possibly after an application of move V to the corresponding signed graph, see Figure 12) to Reidemeister-I style loops. These do not affect $F_L(a, z)$.

The theorem follows immediately. □

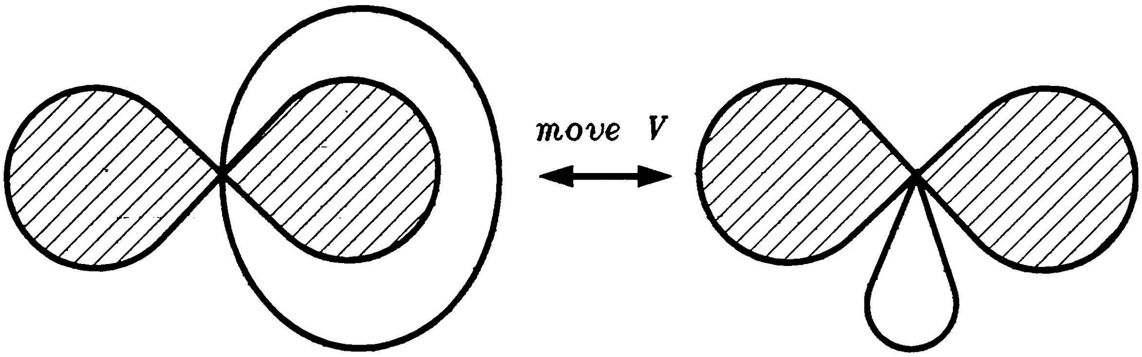


FIGURE 12

I conclude this section with an interesting observation. Gordon and Litherland [6] defined a signature σ_L for an unoriented link (differing from the classical signature of an oriented link by a term which is essentially the total linking number) and showed that it may be calculated from the signature of a Goeritz matrix by using a correction term calculated from the "types" of crossings in the associated diagram (Figure (13)).

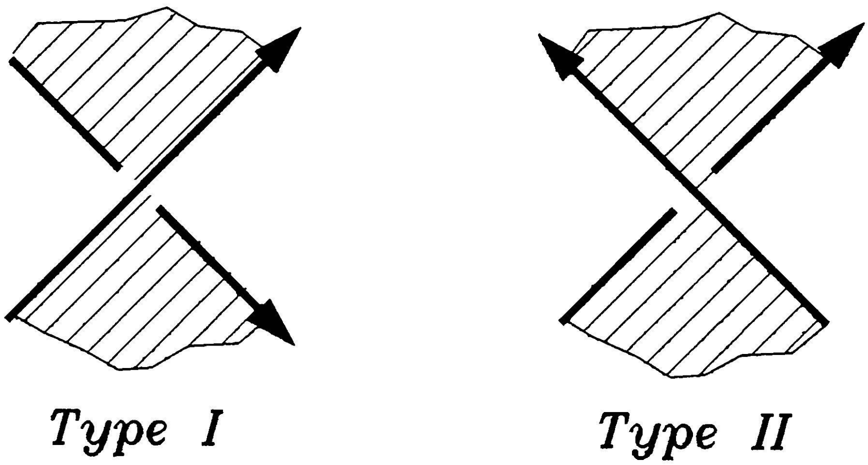


FIGURE 13

Their expression for σ_L is

$$\sigma(L) = \sigma(G) - \sum_{\mathrm{II}} \xi(c)\,, \tag{26}$$

the sum being taken over all crossings of type II where a single component of the link meets itself (at such crossings the type can be determined without ascribing an orientation to the link; for oriented links one sums over all crossings of type II to obtain the classical signature). But following through the same argument as above for the $\tilde{F}$-polynomial, we can show that the proper writhe of a "reduced" diagram (i.e. one with neither loops nor isthmuses in the corresponding signed graph) is a function of the Goeritz matrix (One uses precisely the same reasoning: Examine the effects of the moves of Corollary 6 and show that only the sums of signs of crossings joining adjacent regions are relevant). So given a Goeritz matrix for a diagram $\mathcal{D}$, the proper writhe of the diagram can be used to calculate the number of loops and isthmuses in the corresponding signed graph. Hence Gordon and Litherland's correction term $\sum_{\mathrm{II}} \xi(c)$ can be calculated from the Goeritz matrix and the proper writhe of the diagram, and conversely the proper writhe can be obtained from the Goeritz matrix and this term. So in the presence of the proper writhe of a diagram, the Goeritz matrix can be used to calculate the (unoriented link) signature σ_L.

Now, Thistlethwaite in [21], and Murasugi in [17] have proved

LEMMA 9. *The (proper) writhe of an alternating reduced diagram of a link L is an invariant of the link.* □

from which follows:

LEMMA 10. *The signature of an alternating unoriented link is a function of any Goeritz matrix for that link.* □

This should be compared with the result, also in [21] and [17]:

LEMMA 11. *The (classical) signature of an alternating link is a function of the F-polynomial of the link.* □

Theorem 4 raises the interesting question of what relation there is between $F_L(a, z)$ and the quadratic forms represented by Goeritz matrices. In particular, can either of the last two results be improved to cover non-alternating links?

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