IX. Relations with mathematical physics

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2. Similar remarks apply to the Homfly model of section 3. In the case of the Yang-Baxter model for the Homfly polynomial given in section 7, it is easy to see that the highest z-degree is $\rho(K)$ when K is positive — this time by constructing an appropriate spin state.

3. Thistlethwaite [89] proves that the writhe $w(K)$ is an ambient isotopy invariant for K alternating and reduced. It would be useful to see a proof of this result using the skein model for D_K (section 4).

4. The Alexander polynomial Δ_K is given by the formula

$$\Delta_K(t) \doteq \nabla_K(\sqrt{t}-1/\sqrt{t})$$
$$= \sum_{|L|=1} (-1)^{t-(L)}(\sqrt{t}-1/\sqrt{t})^{t(L)}$$

where $\doteq$ denotes equality up to sign and powers of t. One knows ([23]) that if K bounds a smooth disk in the upper 4-space $((x, y, z, t)$ with $t>\theta)$ then

$$\Delta_K(t) \doteq f(t)f(t^{-1})$$

for some polynomial $f(t)$. Can this fact be deduced directly from the skein model or from the FKT model? A solution should generalize to give new information about the full skein polynomial behaviours on slice links.

IX. Relations with mathematical physics

I have deliberately included a description of the Yang-Baxter models in this paper in order to raise the question of the relation of the skein models to mathematical physics. The Yang-Baxter models can be regarded as averages of scattering amplitudes over all possible spin states — hence as discrete Feynman integrals, or as partition functions for two-dimensional statistical mechanics models. The FKT model for the Conway polynomial can be seen [57] as the low temperature limit of a partition function of a generalized Potts model.

Meta-time

If we interpret the FKT model or the skein models in a particle interaction framework, then a curious and interesting issue arises:

Think of a particle moving forward and backward in "time" on a given universe. The "same" particle may traverse a given site (crossing) twice.

Locally this appears as an interaction of two distinct particles. But in the mathematical trajectory one of these particles came through the site "first" (all dependent on the basepoint or template). And the "way" the particle comes through first may make it particle or anti-particle on this "first pass".

In the skein and FKT models the local vertex weights depend upon this aspect of global trajectory structure. The skein models care which strand is the locus of the "first pass" and the FKT model needs to know whether the first pass is a particle (with the local time-line) or an anti-particle (against the local time-line). Remarkably and fortunately for knot theory, the total summation over all trajectories (denoted $<K>$ in both models) is independent of the choice of template. Thus the first objection to considering "meta-time discriminations" disappears in the averaging. These models solve an issue of invariance that must arise for scattering amplitudes that include the issue of meta-time ordering. It would be very useful to see a parallel problem on this theme in mathematical physics proper.

Into the Third Dimension

Find definitions of the invariants that are intrinsically three-dimensional — that do not depend upon the use of a diagram. This is solved classically for the Alexander polynomial (see [12], [14], [43], [82] for various accounts). The problem remains open in its full generality for the other skein polynomials.

Witten [99] has suggested a definition as the integral over all gauge connections of the trace of the holonomy of the connection — along the knot. The measure in the integral is weighted by the Chern-Simons Lagrangian. Witten's work may well be the desired answer to the question of an intrinsic definition. If so, then there arises a host of questions about the relationship of the holonomy and state models approaches. (Compare [71], [83], [96].)

In the case of the skein models, the relationship is direct. Once the exchange relation is proved from properties of the holonomy, the skein model follows just as we have constructed it here. The question about "meta-time" is then transposed to the more concrete matter of keeping track of orders of computation of holonomies on circuits in the diagram.

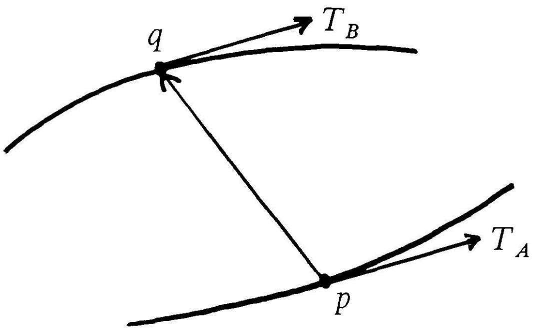
More generally, the possibility of intrinsic models raises the question of the *nature of a crossing in a link diagram.* From the point of view of the plane, a crossing is where the plane curve interacts with itself or with another component. From the point of view of space, the crossing is two distinct points in the space-curve that are projected to the same point in the plane for a given direction of projection.

The relation of these two points of view is seen in sharp relief in the Gauss definition of the linking number of two curves A and B in three-space:

$$\mathrm{lk}(A, B) = \iint_{A \times B} \mathscr{G} ds_A ds_B$$

Here the Gauss Kernel $\mathscr{G}$ is given by the formula

$$\mathscr{G} = e \cdot (T_A \times T_B)/\| e \|^3$$



where e is a direction vector between two points, p and q, one from each curve. T_A and T_B are unit tangent vectors to the curves at these points.

Note that the Gauss kernel vanishes whenever the direction vector and the two tangent vectors occupy the same plane. Thus if we use the Gauss definition to calculate the linking number for a nearly planar diagram, then the result is a sum of vertex contributions (the neighborhoods of the vertices are the only contributors to the integral). (See [27], [97].)

It appears that the Gauss kernel holds a clue to how the state models (also built from local vertex contributions) can be defined three-dimensionally. In such a model, states would be defined directly on the space curve, and vertex weights would be replaced by weights of self-interaction of the knot, modulated by the Gauss kernel.

To see the approximate form of such a model, suppose that we are generalizing the Yang-Baxter model for pl diagrams. Rewrite the model so that it is in exponential form

$$\langle K \rangle = \sum_{\sigma} e^{[K|\sigma]}$$

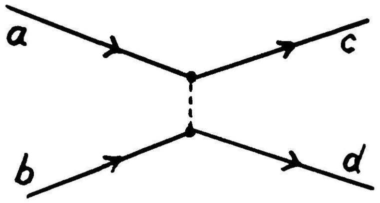
Then sums of vertex weights appear in the exponents. These divide into weights from 2-vertices (corresponding to curvature) and weights from interactions. The interaction weights depend upon angle and local spins — all information that is available on projecting in a given direction e.

This suggests the form of a model as

$$<K> = \int_{\sigma} d\sigma \exp\left(\iint_{K\times K} [p, q \mid \sigma]\, \mathcal{G} + \int_{K} [p \mid \sigma]\right)$$

where the problem of taking the limit over subdivisions of the space curve, and the definition of the limiting state space is certainly unsolved.

The proposed model is designed as a generalization of the planar models. The crossings are replaced by pairs of points on the curve, and the Gauss kernel appears, as in the linking number. In a piecewise-linear approximation to the model, the space curve is divided into straight-line segments. A state assigns a spin to each segment. If spin remains unchanged at a vertex, then the vertex contributes a simple angular term, as in the planar case. *If spin changes at a vertex, then this vertex must be paired up with another vertex so that the pair can be regarded as under-going a spin preserving interaction.*



$$a + b = c + d$$

This is the generalization of the crossing in the planar case. A state is admissible if it is configured with such self-interactions allowing spin conservation. The three-dimensional approximation sums over all such admissible states.

Appendix on state model formalism

This appendix is a short note on the formalism I use for expressing state models.

The bracket polynomial [41] is defined via equations of the form

$$\langle \mathrm{O}K \rangle = d \, \langle K \rangle$$