

# 1. Sets of bounded height

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## III. MORDELL'S CONJECTURE

Suppose  $L$  is a field of characteristic zero of finite type over a relatively algebraically closed subfield  $K$ .

**THEOREM 3.1 (Manin).** *Suppose  $C$  is a curve of genus at least 2 defined over  $K$ . Suppose  $C(L)$  is infinite, then there exists a curve  $C_0$  defined over  $K$  such that  $C_0 \times_K L \cong C$  and  $C(K)$  minus the image of  $C_0(K)$  under this isomorphism is finite.*

We can translate this into

**THEOREM 3.1 (BIS).** *Suppose  $S$  is a variety defined over  $K$  and suppose  $C \rightarrow S$  is a smooth proper curve of genus at least 2 over  $S$ . Suppose  $C(S)$  is infinite, then there exists a curve  $C_0$  defined over  $K$  such that  $C_0 \times_K S \cong C$  and  $C(S)$  minus the image of  $C_0(K)$  under this isomorphism is finite.*

*Remarks.* First, it is possible to reduce this by standard arguments to the case in which  $S$  is a smooth affine curve over  $K$  and so we will suppose this to be the case. Second, if we can prove that  $C_0 \times_K X \cong C$  for some  $C_0$  defined over  $K$ , (i.e. that  $C$  is a constant family) then this is de Franchis' theorem which is proven in Lang's *Fundamentals of Diophantine Geometry*. Hence to prove this theorem all we have to do is show that if  $C(S)$  is infinite then  $C$  is a constant family of curves.

## 1. SETS OF BOUNDED HEIGHT

In this section we will either recall or derive the properties of heights needed in the sequel.

Let  $f: X \rightarrow S$  be a smooth projective morphism of varieties over  $K$  a field of characteristic zero. Corresponding to a projective embedding of  $X$  over  $S$ , there exists a function  $h: X(S) \rightarrow \mathbf{R}$  called a logarithmic height. (For a reference, see ([L-FD] Chapter 3, §3). If the logarithmic height of a subset of  $X(S)$  is bounded with respect to one projective embedding, it is bounded with respect to all (See [L] Prop. 1.7, Chapt. 4). We will call such a set a set of bounded height and a set of points which is not of bounded height, a set of unbounded height. We will need several properties of such sets. If  $g: X' \rightarrow X$  is a morphism of projective schemes over  $S$  which is finite onto its image, then the inverse image of a set of bounded height in  $X(S)$  is a set of bounded height

in  $X'(S)$ . Suppose  $X$  is an Abelian scheme over  $S$  and  $R$  is the subgroup of  $X(S)$  consisting of constant sections of  $X/S$ . Let  $s \in X(S)$ . Then the set  $s + R$  is a set of bounded height.

LEMMA 3.1.1 (Manin). *Suppose  $E$  is a finite dimensional  $K$  vector subspace of  $K(C)$ . Then the set*

$$T = \{s \in C(S) : \exists k \neq 0 \in E \text{ such that } s*k = 0\}$$

*has bounded height.*

*Proof.* Without loss of generality we may increase  $E$  to suppose that the rational map  $g: C \rightarrow \mathbf{P}_K(E)$  given on points by  $x \rightarrow (e \in E \rightarrow e(x))$  is birational onto its image (note:  $g$  is actually a morphism on the complement of the polar locus of  $E$ ). It follows that  $g$  induces an embedding of the generic fiber of  $C/S$  into  $\mathbf{P}_{K(S)}(E \otimes K(S))$ . Let  $h$  denote the logarithmic height with respect to this embedding. It follows that if  $s \in C(S)$ ,  $g \circ s$  is constant or  $g \circ s$  has degree one. In the former case  $h(s)$  is zero and the degree of the Zariski closure of  $g \circ s(S)$  in  $\mathbf{P}(E)$  in the latter.

Now if  $s \in T$ , and  $g \circ s$  is not constant, it follows that the Zariski closure of  $g \circ s(S)$  is a component of a hyperplane section of the Zariski closure of  $g(C)$ . Hence,  $h(s)$  is less than or equal to the degree of the Zariski closure of  $g(C)$ . This proves the lemma.  $\square$

The key property about heights we will need is:

THEOREM 3.1.2. *Suppose  $C \rightarrow S$  is as in the above theorem. If  $C(S)$  contains an infinite set of bounded height then  $C$  is a constant family.*

(See Corollary 2.2, Chapter 8 of [L-FD].)

Hence all we need prove is that the elements of  $C(S)$  have bounded height.

## 2. LANG-SIEGEL TOWERS

Suppose the genus of  $C$  is at least 1. Suppose  $T$  is an infinite subset of  $C(S)$ .

PROPOSITION 3.2.1. *There exists a projective system of curves*

$(\{C_n\}, \{h_{m,n}\}), m, n \in \mathbf{Z}_{>0}$  and  $n \leq m$ , over  $K$  such that

- (i)  $C_1 = C$ ,
- (ii)  $h_{m,n}: C_m \rightarrow C_n$  is étale,