3. Sections of a family and extensions of connections

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connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that S is an affine curve over K and $Z = \varnothing$. Then the short exact sequence (2.2) becomes

$$0 \to f^*\Omega^1_S \otimes \Omega^{\cdot}_{X/S}(-1) \to \Omega^{\cdot}_{X/S} \to \Omega^{\cdot}_{X/S} \to 0\ .$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$(2.3) \qquad \ldots \to H^i_{DR}(X/S) \xrightarrow{\nabla} \Omega^1_S \otimes H^i_{DR}(X/S) \to H^{i+1}_{DR}(X) \to H^{i+1}_{DR}(X/S) \xrightarrow{\nabla} \ldots$$

3. Sections of a family and extensions of connections

Suppose now S is a smooth connected affine curve over a field K of characteristic zero and $f: X \to S$ is a smooth proper morphism of schemes over K, with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose Z is a closed subscheme of X finite over S. Suppose the normalization $n: \tilde{Z} \to Z$ of Z is smooth over S. After repeated blowing ups at closed points we find a scheme $m: \tilde{X}' \to X$, which contains $\tilde{Z}$ and is such that the restriction of m to $\tilde{Z}$ is n. Let $\tilde{X}$ equal the complement in $\tilde{X}'$ of the singular locus of $\tilde{X}'/S$. This locus is a closed subscheme of $\tilde{X}'$ disjoint from $\tilde{Z}$. The long exact sequence 2.1 becomes

$$(3.1) \qquad 0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(\tilde{Z}/S, \tilde{Z}) \to H^1_{DR}(\tilde{X}/S) \to 0$$

Let H denote the pullback of $H^1_{DR}(\tilde{X}/S, \tilde{Z})$ by means of the horizontal monomorphism from $H^1_{DR}(X/S)$ into $H^1_{DR}(\tilde{X}/S)$. We claim that H is independent of the choice of $\tilde{X}$. Indeed, there exists a non-empty affine open subscheme S' of S such that the map from $\tilde{X} \times_S S'$ to $X' = : X \times_S S'$ is an isomorphism. If $Z' = Z \times_S S'$, then Z' is smooth over S' and it is easy to see that $H \otimes K[S'] \cong H^1_{DR}(X'/S', Z')$. Hence H is an extension of the connection $H^1_{DR}(X'/S', Z')$ on S' to a connection on S. Since such an extension is unique if it exists, it follows that H is independent of the choice of $\tilde{X}$ and so we set $H^1_{DR}(X/S, Z) = H$. We obtain from the previous exact sequence, a natural exact sequence

$$0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0\ .$$

For a section s of X/S, we will also use s to denote the induced reduced closed subscheme $s(S)$ of X when convenient. Now suppose s and t are two distinct sections of X/S. Let $Z = s \cup t$. Then $\tilde{Z}$, the normalization of Z, is just two disjoint copies of S and so is étale over S. (The sections s and t

induce maps from S to $\tilde{Z}$ which we denote by the same names.) The map $t^* - s^*: K[\tilde{Z}] \to K[S]$ is horizontal, surjective and its kernel is the image of $K[S]$ under the map in (3.1). Hence we obtain a horizontal exact sequence

$$0 \to K[S] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0$$

and so an extension of $H^1_{DR}(X/S)$ by the trivial connection. We let $E(s, t)$ denote this extension if $s \neq t$ and $E(s, s)$ denote the trivial extension of $H^1_{DR}(X/S)$ by $K[S]$. We call the class of $E(s, t)$ in $\mathrm{Ext}(H^1_{DR}(X/S), K[S])$ $M(s, t)$.

PROPOSITION 1.3.1. *Suppose r, s, t are sections of X/S. Then*

$$M(r, t) = M(r, s) + M(s, t) .$$

In particular, $M(r, s) = -M(s, r)$.

Proof. If r, s and t are not distinct the proposition is obvious from the definitions. Therefore suppose that r, s and t are distinct. If T is a subset of $\{r, s, t\}$ let $Z_T = \bigcup_{u \in T} u$. Either by replacing X by $\tilde{X}$ or by shrinking S and using Corollary 1.1.3 we may assume that $Z_{\{r,s,t\}}$ is étale over S. Let $\mathscr{F}_T$ denote the complex $\Omega^{\cdot}_{X/S, Z_T}$. We set $H(T) = H^1_{DR}(X/S, Z_T)$. Then from the exact sequence of complexes

$$0 \to \mathscr{F}_{\{r,s,t\}} \to \mathscr{F}_{\{r,s\}} \otimes \mathscr{F}_{\{s,t\}} \to \mathscr{F}_{\{s\}} \to 0$$

(where the first map is the diagonal and the last is the difference) we obtain an exact sequence

$$0 \to H(r, s, t) \to H(r, s) \oplus H(s, t) \to H(s)$$

moreover, $H(s) \cong H^1_{DR}(X/S)$ and the last map is the difference of the maps from $H(r, s)$ and from $H(s, t)$ to $H^1_{DR}(X/S)$ (and is, in particular, a surjection).

Next from the exact sequence of complexes

$$0 \to \mathscr{F}_{\{r,s,t\}} \to \mathscr{F}_{\{r,t\}} \to \mathscr{S} \to 0$$

where $\mathscr{S}$ is the complex $(\mathscr{F}_{\{r,t\}}/\mathscr{F}_{\{r,s,t\}} \to 0 \to \ldots) \cong (K[S] \to 0 \to \ldots)$ we obtain an exact sequence

$$0 \to K[S] \to H(r, s, t) \to H(r, t) \to 0$$

Moreover the first map is the composition of the map from $K[Z_{\{r,s,t\}}]$ into $H(r, s, t)$ and the map h from $K[S]$ into $K[Z_{\{r,s,t\}}]$ characterized by $r^*h(f) = t^*h(f) = 0$ and $t^*h(f) = f$. It follows from this that $H(r, t)$ is the

Baer sum of $H(r,s)$ and $H(s,t)$. Since all the maps discussed above are horizontal this statement is true on the level of connections as well. This proves the proposition. □

Suppose X' is a smooth scheme over S and $g: X' \to X$ is an S-morphism. Then the natural map $g^*: H^1_{DR}(X/S) \to H^1_{DR}(X/S)$ induces a natural map $g^*: \mathrm{Ext}(H^1_{DR}(X'/S), K[S]) \to \mathrm{Ext}(H^1_{DR}(X/S), K[S])$. By the naturality of all our constructions we have:

PROPOSITION 1.3.2. *Suppose X'/S has geometrically connected fibers and s and t are two sections of X'/S. Then*

$$M(g\circ s, g\circ t) = g^*M(s,t) .$$

Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Then

$$(\Omega^{\cdot}_{X/S}, d_{X/S}) \cong K[S] \otimes (\Omega^{\cdot}_{X_0/K}, d_{X_0/K})$$

and so in particular,

$$H^1_{DR}(X/S) \cong K[S] \otimes H^1_{DR}(X_0/K)$$

and the Gauss-Manin connection

$$\nabla: H^1_{DR}(X/S) \to \Omega^1_S \otimes_{K[S]} H^1_{DR}(X/S)$$

is (d, id). If $H = H^1_{DR}(X/S)$, it follows from this that

$$\mathrm{Ext}(H, K[S]) \cong H^1(\check{H}, \check{\nabla}) \cong \mathrm{Hom}_K(H^1_{DR}(X_0/K), H^1_{DR}(S/K)) .$$

Explicitly, this last isomorphism can be described as follows:

$$\text{if} \quad h \in \mathrm{Hom}(H, \Omega^1_S) \cong \Omega^1_S \otimes \check{H} ,$$

then $h \bmod \check{\nabla}\check{H}$ goes to the map $(\omega \in H^1_{DR}(X_0/K) \to h(1 \otimes \omega) \bmod dK[S])$.

PROPOSITION 1.3.3. *Suppose X_0 is a smooth connected scheme over K and $X = S \times_K X_0$. Suppose u and v are two morphisms from S to X_0 and $s = (id, u)$ and $t = (id, v)$. Then $M(s,t)$ is $v^* - u^*$ as an element of* $\mathrm{Hom}_K(H^1_{DR}(X_0/K), H^1_{DR}(S/K))$.

Proof. We may suppose that $s \cap t = \varnothing$. Let $Z = s \cup t$. Suppose $h: H^1_{DR}(X/S) \to H^1_{DR}(X/S, Z)$ is a section. Let $(\{\omega_U\}, \{f_{U,V}\})$ be a one-hypercocycle for $(\Omega^{\cdot}_{X_0/K}, d_{X_0/K})$ and $[\omega]$ the image the class of $1 \otimes (\{\omega_U\}, \{f_{U,V}\})$ in $H^1_{DR}(X/S)$. Then $\nabla[\omega] = 0$. We wish to compute $\nabla h([\omega]) - h(\nabla[\omega]) = \nabla h([\omega])$. We will abuse notation and identify ω_U with $1 \otimes \omega_U$ in $\Omega^1_X(U)$ and $f_{U,V}$ with $1 \otimes f_{U,V}$ in $\mathscr{O}_X(U \cap V)$. Let $\bar{\omega}_U$ denote the image of ω_U in $\Omega^1_{X/S}(U)$. Then $h([\omega])$ is the class of

$$(\{\bar{\omega}_U - d_{X/S}g_U\}, \{f_{U,V} - (g_U - g_V)\})$$

for some one-chain $\{g_U\}$ with coefficients in $\mathscr{O}_X$ such that

$$s^*f_{U,V} = u^*f_{U,V} = s^*(g_U - g_V) \quad \text{and} \quad t^*f_{U,V} = v^*f_{U,V} = t^*(g_U - g_V)\ .$$

Let $\eta_U = \omega_U - dg_U$. Now

$$s^*\eta_U - s^*\eta_V = s^*df_{U,V} - s^*d(g_U - g_V) = 0$$

by the conditions that $\{g_U\}$ must satisfy and the fact that $(\{\omega_U\}, \{f_{U,V}\})$ is a hypercocycle. Similarly, $t^*\eta_U - t^*\eta_V = 0$. Let η_s and η_t be the the elements of Ω_S^1 determined by the cocycles $\{s^*\eta_U\}$ and $\{t^*\eta_U\}$ respectively.

Now to compute $\nabla h([\omega])$ we must lift $\bar{\omega}_U - d_{X/S}g_U$ to a section of $\Omega_{X,Z}^1$. Let $e_{s,U}$ and $e_{t,U}$ be elements of $\mathscr{O}_X(U)$ such that $s^*e_{s,U} = 1\, t^*e_{t,U} = 0$, $t^*e_{t,U} = 1$ and $s^*e_{t,U} = 0$. These elements exist since Z is étale over S. Then $\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)$ is such a lifting. To compute $\nabla h([\omega])$ we must take the hyper-coboundary of $(\{\eta_U - (e_{s,U}\eta_s + e_{t,U}\eta_t)\}, \{f_{U,V} - (g_U - g_V)\})$. It is

$$(\{\eta_s \otimes d_{X/S}e_{s,U} + \eta_t \otimes d_{X/S}e_{t,U}\}, \{\eta_s \otimes (e_{s,U} - e_{s,V}) + \eta_t \otimes (e_{t,U} - e_{t,V})\}, 0)\ .$$

The class of this hypercocycle is the image of

$$\eta_t - \eta_s \in \Omega_S^1 \quad \text{in} \quad \Omega_S^1 \otimes H_{DR}^1(X/S, Z)$$

(recall that we've determined a map of $K[S]$ into $H_{DR}^1(X/S, Z)$). Hence $\nabla h([\omega]) = \eta_t - \eta_s$.

The proposition now follows from the fact that

$$(\{\eta_s + ds^*g_U\}, \{s^*g_U - s^*g_V\}) = u^*(\{\omega_U\}, \{f_{U,V}\})$$

and

$$(\{\eta_t + dt^*g_U\}, \{t^*g_U - t^*g_V\}) = v^*(\{\omega_U\}, \{f_{U,V}\})\ . \qquad \square$$

COROLLARY 1.3.4. *If, in the above, u and v are constant, then* $M(s, t) = 0$.

4. ABELIAN SCHEMES

Suppose now that A is an Abelian scheme over S. Let $m: A \times_S A \to A$ be the addition law and e the zero section. For $s, t \in A(S)$, let $M(s) = M(e, s)$ and $s + t = m(s, t)$.

THEOREM 1.4.1. *The map M from $A(S)$ to* $\mathrm{Ext}(H_{DR}^1(A/S), K[S])$ *is a homomorphism.*