

2. The Gauss-Manin connection

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since $\pi(s(v)) = v$ and $\pi \nabla(e) = \nabla_H(\pi(e))$. The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e))) = 0. \quad \square$$

Suppose W is an $\mathcal{O}_S$ submodule of H . We let $[W]$ denote the smallest subconnection of H containing W .

2. THE GAUSS-MANIN CONNECTION

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If $\mathcal{S}^\bullet$ is a complex, $\mathcal{S}^\bullet(k)$ will denote the complex obtained from $\mathcal{S}^\bullet$ by setting $\mathcal{S}^i(k) = \mathcal{S}^{i+k}$. For any scheme Y over K will let $K[Y]$ denote $\Gamma(\mathcal{O}_Y)$.

Suppose S is a smooth connected affine scheme over K . Suppose $f: X \rightarrow S$ is a smooth morphism, Z is a closed subscheme of X , smooth over S . Suppose T is either $\text{Spec}(K)$ or S . Then we define the subcomplex $\Omega_{X/T, Z}^\bullet$ of $\Omega_{X/T}^\bullet$ by the exactness of the sequence.

$$0 \rightarrow \Omega_{X/T, Z}^\bullet \rightarrow \Omega_{X/T}^\bullet \rightarrow \Omega_{Z/T}^\bullet \rightarrow 0.$$

When $T = \text{Spec}(K)$ we drop it from the notation. It follows that $\Omega_{X/S, Z}^i = \Omega_{X/S}^i$ for $i > \dim_S Z$. Note that $\Omega_{X, Z}^0 = \Omega_{X/S, Z}^0$ is the sheaf of ideals of Z on X . We define $H_{DR}^i(X/S, Z)$ to be the i -th hypercohomology group of the complex $\Omega_{X/S, Z}^\bullet$. We set $H_{DR}^i(X/S) = H_{DR}^i(X/S, \emptyset)$. If X is affine, then $H_{DR}^i(X/S, Z)$ is the i -th cohomology group of the complex of $K[S]$ modules $\Gamma(\Omega_{X/S, Z}^\bullet)$. If X is affine, K has characteristic zero and U is a dense open subscheme of X then the natural map from $H_{DR}^i(X/S, Z)$ to $H_{DR}^i(U/S, U \cap Z)$ is an injection.

From the last short exact sequence with $T = S$, we obtain a long exact sequence

$$(2.1) \quad \dots \rightarrow H_{DR}^{i-1}(Z/S) \rightarrow H_{DR}^i(X/S, Z) \rightarrow H_{DR}^i(X/S) \rightarrow \dots$$

The Gauss-Manin connection $\nabla: H_{DR}^i(X/S, Z) \rightarrow \Omega_S^1 \otimes H_{DR}^i(X/S, Z)$ is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

$$(2.2) \quad 0 \rightarrow f^* \Omega_S^1 \otimes \Omega_{X/S, Z}^\bullet(-1) \rightarrow \Omega_{X/S, Z}^\bullet / f^* \Omega_S^2 \otimes \Omega_X^\bullet(-2) \rightarrow \Omega_{X/S, Z}^\bullet \rightarrow 0$$

(which is exact because X and Z are smooth over S). It is an integrable connection. If K has characteristic zero and f is surjective and has geometrically connected fibers, then $H_{DR}^0(X/S) = K[S]$ and the Gauss-Manin

connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that S is an affine curve over K and $Z = \emptyset$. Then the short exact sequence (2.2) becomes

$$0 \rightarrow f^* \Omega_S^1 \otimes \Omega_{X/S}^1(-1) \rightarrow \Omega_{X/S}^1 \rightarrow \Omega_{X/S}^1 \rightarrow 0.$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$(2.3) \quad \dots \rightarrow H_{DR}^i(X/S) \xrightarrow{\nabla} \Omega_S^1 \otimes H_{DR}^i(X/S) \rightarrow H_{DR}^{i+1}(X) \rightarrow H_{DR}^{i+1}(X/S) \xrightarrow{\nabla} \dots$$

3. SECTIONS OF A FAMILY AND EXTENSIONS OF CONNECTIONS

Suppose now S is a smooth connected affine curve over a field K of characteristic zero and $f: X \rightarrow S$ is a smooth proper morphism of schemes over K , with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose Z is a closed subscheme of X finite over S . Suppose the normalization $n: \tilde{Z} \rightarrow Z$ of Z is smooth over S . After repeated blowing ups at closed points we find a scheme $m: \tilde{X}' \rightarrow X$, which contains $\tilde{Z}$ and is such that the restriction of m to $\tilde{Z}$ is n . Let $\tilde{X}$ equal the complement in $\tilde{X}'$ of the singular locus of $\tilde{X}'/S$. This locus is a closed subscheme of $\tilde{X}'$ disjoint from $\tilde{Z}$. The long exact sequence 2.1 becomes

$$(3.1) \quad 0 \rightarrow K[S] \rightarrow K[\tilde{Z}] \rightarrow H_{DR}^1(\tilde{Z}/S, \tilde{Z}) \rightarrow H_{DR}^1(\tilde{X}/S) \rightarrow 0$$

Let H denote the pullback of $H_{DR}^1(\tilde{X}/S, \tilde{Z})$ by means of the horizontal monomorphism from $H_{DR}^1(X/S)$ into $H_{DR}^1(\tilde{X}/S)$. We claim that H is independent of the choice of $\tilde{X}$. Indeed, there exists a non-empty affine open subscheme S' of S such that the map from $\tilde{X} \times_S S'$ to $X' = X \times_S S'$ is an isomorphism. If $Z' = Z \times_S S'$, then Z' is smooth over S' and it is easy to see that $H \otimes K[S'] \cong H_{DR}^1(X'/S', Z')$. Hence H is an extension of the connection $H_{DR}^1(X'/S', Z')$ on S' to a connection on S . Since such an extension is unique if it exists, it follows that H is independent of the choice of $\tilde{X}$ and so we set $H_{DR}^1(X/S, Z) = H$. We obtain from the previous exact sequence, a natural exact sequence

$$0 \rightarrow K[S] \rightarrow K[\tilde{Z}] \rightarrow H_{DR}^1(X/S, Z) \rightarrow H_{DR}^1(X/S) \rightarrow 0.$$

For a section s of X/S , we will also use s to denote the induced reduced closed subscheme $s(S)$ of X when convenient. Now suppose s and t are two distinct sections of X/S . Let $Z = s \cup t$. Then $\tilde{Z}$, the normalization of Z , is just two disjoint copies of S and so is étale over S . (The sections s and t