2. The Gauss-Manin connection

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since $\pi(s(v)) = v$ and $\pi\nabla(e) = \nabla_H(\pi(e))$. The lemma now follows from

$$(t \circ \nabla_G^\vee)(w)(e) = \nabla_G^\vee(w)(e - s(\pi(e)) = 0 \,. \quad \square$$

Suppose W is an $\mathscr{O}_S$ submodule of H. We let [W] denote the smallest subconnection of H containing W.

2. The Gauss-Manin connection

Here we will recall the definition and some basic properties of the Gauss-Manin connection which we will need in this paper. For more details see [K-O]. If $\mathscr{S}^\cdot$ is a complex, $\mathscr{S}^\cdot(k)$ will denote the complex obtained from $\mathscr{S}^\cdot$ by setting $\mathscr{S}^i(k) = \mathscr{S}^{i+k}$. For any scheme Y over K will let $K[Y]$ denote $\Gamma(\mathscr{O}_Y)$.

Suppose S is a smooth connected affine scheme over K. Suppose $f: X \to S$ is a smooth morphism, Z is a closed subscheme of X, smooth over S. Suppose T is either Spec(K) or S. Then we define the subcomplex $\Omega^\cdot_{X/T,Z}$ of $\Omega^\cdot_{X/T}$ by the exactness of the sequence.

$$0 \to \Omega^\cdot_{X/T,Z} \to \Omega^\cdot_{X/T} \to \Omega^\cdot_{Z/T} \to 0 \,.$$

When $T = \mathrm{Spec}(K)$ we drop it from the notation. It follows that $\Omega^i_{X/S,Z} = \Omega^i_{X/S}$ for $i > \dim_S Z$. Note that $\Omega^0_{X,Z} = \Omega^0_{X/S,Z}$ is the sheaf of ideals of Z on X. We define $H^i_{DR}(X/S, Z)$ to be the i-th hypercohomology group of the complex $\Omega^\cdot_{X/S,Z}$. We set $H^i_{DR}(X/S) = H^i_{DR}(X/S, \varnothing)$. If X is affine, then $H^i_{DR}(X/S, Z)$ is the i-th cohomology group of the complex of $K[S]$ modules $\Gamma(\Omega^\cdot_{X/S,Z})$. If X is affine, K has characteristic zero and U is a dense open subscheme of X then the natural map from $H^i_{DR}(X/S, Z)$ to $H^i_{DR}(U/S, U \cap Z)$ is an injection.

From the last short exact sequence with $T = S$, we obtain a long exact sequence

(2.1) $$\ldots \to H^{i-1}_{DR}(Z/S) \to H^i_{DR}(X/S, Z) \to H^i_{DR}(X/S) \to \ldots$$

The Gauss-Manin connection $\nabla: H^i_{DR}(X/S, Z) \to \Omega^1_S \otimes H^i_{DR}(X/S, Z)$ is the boundary map in the long exact sequence obtained by taking hypercohomology of the short exact sequence of complexes:

(2.2) $$0 \to f^*\Omega^1_S \otimes \Omega^\cdot_{X/S,Z}(-1) \to \Omega^\cdot_{X/S,Z}/f^*\Omega^2_S \otimes \Omega^\cdot_X(-2) \to \Omega^\cdot_{X/S,Z} \to 0$$

(which is exact because X and Z are smooth over S). It is an integrable connection. If K has characteristic zero and f is surjective and has geometrically connected fibers, then $H^0_{DR}(X/S) = K[S]$ and the Gauss-Manin connection is the trivial connection on this module. Moreover, it is easy to show that the sequence (2.1) is horizontal with respect to the respective Gauss-Manin connections.

Suppose now that S is an affine curve over K and $Z = \varnothing$. Then the short exact sequence (2.2) becomes

$$0 \to f^*\Omega^1_S \otimes \Omega^{\cdot}_{X/S}(-1) \to \Omega^{\cdot}_{X/S} \to \Omega^{\cdot}_{X/S} \to 0\,.$$

Taking cohomology of this sequence yields the Leray long exact sequence

$$\text{(2.3)} \qquad \ldots \to H^i_{DR}(X/S) \xrightarrow{\nabla} \Omega^1_S \otimes H^i_{DR}(X/S) \to H^{i+1}_{DR}(X) \to H^{i+1}_{DR}(X/S) \xrightarrow{\nabla} \ldots$$

3. Sections of a family and extensions of connections

Suppose now S is a smooth connected affine curve over a field K of characteristic zero and $f: X \to S$ is a smooth proper morphism of schemes over K, with geometrically connected fibers. These assumptions will be in force throughout the remainder of this paper. Suppose Z is a closed subscheme of X finite over S. Suppose the normalization $n: \tilde{Z} \to Z$ of Z is smooth over S. After repeated blowing ups at closed points we find a scheme $m: \tilde{X}' \to X$, which contains $\tilde{Z}$ and is such that the restriction of m to $\tilde{Z}$ is n. Let $\tilde{X}$ equal the complement in $\tilde{X}'$ of the singular locus of $\tilde{X}'/S$. This locus is a closed subscheme of $\tilde{X}'$ disjoint from $\tilde{Z}$. The long exact sequence 2.1 becomes

$$\text{(3.1)} \qquad 0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(\tilde{Z}/S, \tilde{Z}) \to H^1_{DR}(\tilde{X}/S) \to 0$$

Let H denote the pullback of $H^1_{DR}(\tilde{X}/S, \tilde{Z})$ by means of the horizontal monomorphism from $H^1_{DR}(X/S)$ into $H^1_{DR}(\tilde{X}/S)$. We claim that H is independent of the choice of $\tilde{X}$. Indeed, there exists a non-empty affine open subscheme S' of S such that the map from $\tilde{X} \times_S S'$ to $X' =: X \times_S S'$ is an isomorphism. If $Z' = Z \times_S S'$, then Z' is smooth over S' and it is easy to see that $H \otimes K[S'] \cong H^1_{DR}(X'/S', Z')$. Hence H is an extension of the connection $H^1_{DR}(X'/S', Z')$ on S' to a connection on S. Since such an extension is unique if it exists, it follows that H is independent of the choice of $\tilde{X}$ and so we set $H^1_{DR}(X/S, Z) = H$. We obtain from the previous exact sequence, a natural exact sequence

$$0 \to K[S] \to K[\tilde{Z}] \to H^1_{DR}(X/S, Z) \to H^1_{DR}(X/S) \to 0\,.$$

For a section s of X/S, we will also use s to denote the induced reduced closed subscheme $s(S)$ of X when convenient. Now suppose s and t are two distinct sections of X/S. Let $Z = s \cup t$. Then $\tilde{Z}$, the normalization of Z, is just two disjoint copies of S and so is étale over S. (The sections s and t