

7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **26.09.2024**

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PROPOSITION 10. *If I and J are equivalent, reduced, primitive ideals of O_D then*

$$d(J, I) \equiv d(I, J)^{-1} \pmod{\times \eta} .$$

Proof. As I and J are in the same period we have $J = \rho I (\rho \in K^*)$ and $I = \sigma J (\sigma \in K^*)$. As $I = \rho^{-1} J$ we have $\sigma \equiv \rho^{-1} \pmod{\times \eta}$, which proves Proposition 10.

7. COMPARISON OF DISTANCES BETWEEN CORRESPONDING IDEALS IN DIFFERENT ORDERS

Let C be a primitive class of the order O_{Df^2} and let $\theta(C)$ be the image of C by the mapping θ defined in §3. As an application of the concept of distance described in §6, we explain how to define a mapping of the period of C into the period of $\theta(C)$, which approximately preserves distance.

THEOREM 2. *For $D' = Df^2$ let $C \in C_{D'}$ and $\theta(C)$ its image by the surjective homomorphism $\theta: C_{D'} \rightarrow C_D$.*

(i) *There exists a mapping τ from the period of C into the period of $\theta(C)$ such that for I and I' in the period of C we have, for a choice of d modulo units,*

$$(7.1) \quad \frac{d(I, I')}{8f^7 D^{3/2}} < d(\tau(I), \tau(I')) < 8f^7 D^{3/2} d(I, I') .$$

(ii) *When $f = p$ (prime) there exists a mapping σ from the period of C into the period of $\theta(C)$ such that for I and I' in the period of C we have, for a choice d modulo units,*

$$(7.2) \quad \frac{d(I, I')}{2Dp^2} < d(\sigma(I), \sigma(I')) < 2Dp^2 d(I, I') .$$

Proof. Let $I = a[1, \phi] (a > 0)$ and $I' = a'[1, \phi'] (a' > 0)$ be two equivalent, reduced, primitive ideals of a class C of $O_{D'} (D' = Df^2)$ with $\phi = \frac{b + \sqrt{D'}}{2a}$ and $\phi' = \frac{b' + \sqrt{D'}}{2a'}$ reduced. Let $\delta \in K^*$ be such that $I' = \delta I, \delta > 0$.

(i) If $\text{GCD}(a, f) = 1$ we set $I_1 = I$. If $\text{GCD}(a, f) > 1$, from the proof of Lemma 2, we see that there exists an ideal $I_1 = a_1[1, \phi_1] = \rho I$ in C with

$\rho = |x + \bar{\phi}y|$, where x and y are integers such that $a_1 = |ax^2 + bxy - \left(\frac{D' - b^2}{4a}\right)y^2|$, $GCD(a_1, f) = 1$, $GCD(x, y) = 1$, $0 \leq x < f$, $0 \leq y < f$.

As $\phi = \frac{b + \sqrt{D'}}{2a}$ is reduced, we have

$$1 \leq a < \sqrt{D'}, \quad 1 \leq b < \sqrt{D'}, \quad 1 \leq c < \sqrt{D'} \quad \left(c = \frac{D' - b^2}{4a} \right),$$

so that $\phi < \sqrt{D'}$, $|\bar{\rho}| = x + \phi y < f(1 + \sqrt{D'}) < 2f\sqrt{D'}$, and

$$(7.3) \quad 1 \leq a_1 < 2\sqrt{D'} f^2.$$

Also $\phi > 1$, $-1 < \bar{\phi} < 0$, so, as $\rho|\bar{\rho}| = a_1/a$, we have

$$(7.4) \quad \frac{1}{2fD'} < \rho < f.$$

By the way in which we have defined $I_1 = \left[a_1, \frac{b_1 + \sqrt{D'}}{2} \right]$, we have $GCD(a_1, f) = 1$. Appealing to the proof of Theorem 1 (i), we see that there exists an integer b_2 such that $I_1 = \left[a_1, f \left(\frac{b_2 + \sqrt{D}}{2} \right) \right]$.

Similarly there exists an ideal $I'_1 = \left[a'_1, f \left(\frac{b'_2 + \sqrt{D}}{2} \right) \right]$ such that $I'_1 = \rho'I'$

with ρ' satisfying (7.4). Now, by Theorem 1, $J_1 = \left[a_1, \frac{b_2 + \sqrt{D}}{2} \right]$ and

$J'_1 = \left[a'_1, \frac{b'_2 + \sqrt{D}}{2} \right]$ are ideals of $\theta(C)$ such that $J'_1 = \rho'\delta\rho^{-1}J_1$. Applying

the Lagrange reduction process to J_1 and J'_1 , we obtain reduced ideals J and J' , and, by Proposition 7, we have $J = \alpha J_1$, and $J' = \alpha' J'_1$, with (by (7.3))

$$\frac{1}{2f^2\sqrt{D'}} < \frac{1}{a_1} \leq \alpha < 2, \quad \frac{1}{2f^2\sqrt{D'}} < \frac{1}{a'_1} \leq \alpha' < 2.$$

Thus we have $J' = \delta'J$, where $\delta' = \alpha'\rho'\delta\rho^{-1}\alpha^{-1}$ satisfies

$$\frac{\delta}{8f^4D'^{3/2}} < \delta' < 8f^4D'^{3/2}\delta.$$

Setting $J = \tau(I)$ gives the required mapping and proves (7.1).

(ii) When $f = p$ (prime) and p does not divide a , we set $I_1 = I$. If p divides a , we take for I the ideal $a_1[1, \phi_1]$ following I in its period. In this case, as $p \mid a$, from $p^2D = b_1^2 + 4aa_1$, we see that $p \mid b_1$ and so, as $\text{GCD}(a_1, b_1, a) = 1$ we see that p does not divide a_1 . Then, by (2.12), we have $I_1 = \rho I$ with $\rho = \frac{a_1}{a} \phi_1$. Now, by Proposition 5, $\phi_1 = \frac{b_1 + \sqrt{D'}}{2a_1}$ is reduced, so that $1 \leq b_1 < \sqrt{D'}$, and

$$(7.5) \quad 1 \leq a_1 < \sqrt{D'},$$

giving

$$(7.6) \quad 1 \leq \rho < \sqrt{D'}.$$

The rest of the proof follows exactly as in the proof of (i) using (7.5) (resp. (7.6)) in place of (7.3) (resp. (7.4)).

8. GAUSS'S REDUCTION PROCESS

Definition 14. (Half-reduced) A representation $\{a, b\}$ of an ideal I is said to be *half-reduced* if

$$(8.1) \quad 0 < \frac{-b + \sqrt{D}}{2|c|} < 1,$$

where $c = (D - b^2) \mid 4a$.

An ideal I is called *half-reduced* if there exists a half-reduced representation of I .

Clearly, if $\{a, b\}$ is half-reduced, then $b < \sqrt{D}$ and $\{-a, b\}$ is half-reduced.

LEMMA 7. Let I be a primitive ideal of O_D . To each representation $\{a, b\}$ of I corresponds a unique integer q such that the q -neighbour representation $\{a', b'\}$ is half-reduced. The integer b' and the ideal $I' = \left[a', \frac{b' + \sqrt{D}}{2} \right]$ are determined by I . The value of q is

$$(8.2) \quad q = \frac{a}{|a|} \left[\frac{b + \sqrt{D}}{2|a|} \right].$$