L'Enseignement Mathématique, t. 36 (1990), p. 309-320

THE HADAMARD-CARTAN THEOREM IN LOCALLY CONVEX METRIC SPACES

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1. INTRODUCTION

M. Gromov has stated the following theorem, which generalizes the classical Hadamard-Cartan theorem from Riemannian manifolds of non-positive sectional curvature to a much richer class of metric spaces in which sectional curvature need not be defined [Gv1, Gv2]:

THEOREM 1 [Gromov]. *A simply connected, complete, locally convex geodesic space is globally convex; hence any two points are joined by a unique geodesic.*

Our interest in Gromov's theorem arose from the wish to apply it to Riemannian manifolds with boundary (see [ABB]). This note gives a proof of the theorem and relates it to work of A. D. Alexandrov, H. Busemann and S. Cohn-Vossen. In this adaptation of the classical proof of the Hadamard-Cartan theorem to the setting of locally convex geodesic spaces, there are two points that require additional attention: finding an appropriate definition of the exponential map, and showing that it is locally surjective. Additionally, for spaces of curvature bounded above in Alexandrov's sense by $K>0$, we show local surjectivity out to length $\pi/\sqrt{K}$ (Theorem 3); since local injectivity is a consequence of Alexandrov's development method, this fully generalizes the classical estimate on conjugate distance. Finally, we extend the Hadamard-Cartan theorem to locally compact geodesic spaces without conjugate points (Theorem 6). We have subsequently learned that our proof of Theorem 1 parallels a sketch proposed by Gromov in oral lectures; we hope that an exposition of these ideas will be useful.

The class of locally convex geodesic spaces includes, for instance, two-dimensional polyhedral surfaces whose simplices are isometric to simplices in a space of constant nonpositive curvature and for which the total angle at each vertex not on the boundary is at least 2π ([A2]; also see [Gv2, section 4.2] and [Ba] for the n-dimensional analogue). Orbifolds modeled on quotients of nonpositively curved spaces and ramified coverings of nonpositively curved spaces also offer many examples. All these are discussed in Gromov's *Hyperbolic Groups* [Gv2], which contains a far-reaching discussion of various types of hyperbolicity (also see [GLP, chapter 1]). Complete Riemannian manifolds with boundary satisfying appropriate interior and boundary curvature conditions are also examples (see [Gv1] and [ABB]). Note that the examples mentioned here allow bifurcation of geodesics; that is, a maximal extension of a geodesic need not be unique.

The earliest generalization of the Hadamard-Cartan theorem to nonriemannian locally convex spaces seems to be due to Busemann [Bu1], in the context of *G-spaces* (see Remark 2 below). The versions due to Busemann [Bu1, Bu2] and Rinow [R] concern spaces for which geodesics do not bifurcate and are infinitely extendible, and which are locally compact and satisfy domain invariance. Note that all of these hypotheses have been eliminated in Theorem 1. For a simple example of a complete, locally convex geodesic space satisfying none of them, take the metric completion of the simply connected covering of the punctured Euclidean or hyperbolic closed disk. In particular, where Busemann and Rinow use the unique and infinite extendibility of geodesics to define the exponential map on the product of $[0, \infty)$ with a metric sphere around m, here it is defined as the endpoint map on the space of geodesics starting at m, in the uniform metric. Where they use invariance of domain to show that the exponential map is locally surjective, here this is shown to follow from local convexity alone.

Geodesic spaces were first considered by Alexandrov [A1], who defined upper curvature bounds for such spaces and gave a development method for transforming local curvature bounds into global ones under certain conditions (see below). We shall use the following terminology. An *interior* metric space M is one in which the distance between any two points is the infimum of the lengths of curves joining them (where curvelength is defined as usual); the terms *inner* and *tight* have also been used. M is a *geodesic space* if in addition it contains a shortest curve between any two points [Gv2]. (*Length space* has been used with both meanings.) A complete interior space is geodesic if it is locally compact [Bu2, p. 24], but might not be otherwise. For instance, an ellipsoid in Hilbert space, the lengths of whose axes are strictly decreasing and bounded above zero, is complete in the interior metric and yet contains infinitely many pairs of points which cannot be joined by shortest curves (see [Gn]). A sequence of intervals of length $1 + 1/n$ with their left and right endpoints respectively identified is complete and locally convex in the interior metric and contains a pair of points not joined by a shortest curve.

A *geodesic* will always be a locally distance-realizing curve parametrized proportionally to arclength by [0, 1]. A geodesic space is *locally convex* if every point has a neighborhood such that the distance $d(\alpha(t), \beta(t))$ is convex for any two minimizing geodesics α and β in the neighborhood. (Of course, for Riemannian manifolds without boundary this is equivalent to nonpositive sectional curvature; see [BGS].) If m_{pq} denotes the midpoint of a geodesic from p to q, then it is equivalent to say that M is covered by neighborhoods U such that the relation

$$2d(m_{pq}, m_{pr}) \leqslant d(q, r)$$

holds for any three points p, q and r in U and any geodesics in U joining them (such geodesics are unique).

We are very grateful to the referee for examining the paper carefully and suggesting a number of technical improvements.

We also thank the referee for informing us of the chapter [Ba] by W. Ballmann that is to appear in *Sur les Groupes Hyperboliques d'après Gromov* (Ghys, de la Harpe, eds.), and its author for promptly sending us a preprint. In [Ba], the Hadamard-Cartan theorem is proved using the Birkhoff curve-shortening technique; this depends on local compactness, which we avoid by exploiting local convexity. Another distinction is that the notions of exponential map and conjugate point are not introduced in [Ba]. The Hadamard-Cartan theorem is applied in [Ba] to obtain a criterion for the hyperbolicity of certain simply connected polyhedra.

2. Conjugate points

In a given geodesic space, let $\mathbf{G}_m$ be the space of geodesics starting at m, carrying the uniform metric $\mathbf{d}$. Say the point m has *no conjugate points* if the endpoint map on $\mathbf{G}_m$ maps some neighborhood of every γ homeomorphically onto a neighborhood of the endpoint of γ. (In Riemannian manifolds without boundary, this definition is equivalent to the usual one.)

THEOREM 2. *A locally convex, complete geodesic space has no conjugate points.*

Here it is straightforward that the endpoint map is a homeomorphism, and in fact an isometry, from some open neighborhood of every γ onto its image. The question is whether it is surjective; that is, whether locally there always exist geodesics from m that vary continuously with their righthand endpoints.

To show this, especially in the absence of local compactness, seems to require a little care.

Proof. Fix a geodesic γ. The maximum radius of an open ball, such that the distance function is convex between minimizing geodesics with endpoints in the ball, is positive and varies continuously with the center of the ball. Thus the infimum, r, of these radii over all the points of γ is positive. Suppose α_1 and α_2 are geodesics whose distances from γ, namely

$$\mathbf{d}(\alpha_i, \gamma) = \max d(\alpha_i(t), \gamma(t)) ,$$

are less than r. Since convexity is a local property, the distance function $d(\alpha_1(t), \alpha_2(t))$ is convex, and hence the larger of its two endpoint values is $\mathbf{d}(\alpha_1, \alpha_2)$.

Let $P(L)$ be the statement:

> For every subsegment $\bar{\gamma}$ of γ of length at most L, any two points p and q whose respective distances from the endpoints of $\bar{\gamma}$ are less than $r/2$ are joined by a geodesic $\alpha = \alpha(p, q)$ whose distance from $\bar{\gamma}$ is less than $r/2$.

Note that $\alpha(p, q)$ is necessarily unique, and the distance function between any two such geodesics is convex.

Clearly $P(r)$ holds. We claim that $P(3L/2)$ holds if $P(L)$ does. Indeed, suppose $\bar{p}$ and $\bar{q}$ are the left and right endpoints of a subsegment of γ of length at most $3L/2$, and let p_0 and q_0 trisect its length, moving from left to right. Suppose p and q lie within distance $R < r/2$ of the endpoints $\bar{p}$ and $\bar{q}$, respectively. Applying $P(L)$ to $\alpha(\bar{p}, q_0)$ and $\alpha(p_0, \bar{q})$ repeatedly, we define p_i and q_i inductively for $i \geq 1$ by letting p_i be the midpoint of $\alpha(p, q_{i-1})$ and q_i be the midpoint of $\alpha(p_{i-1}, q)$. Convexity ensures that $d(p_{i-1}, p_i)$ and $d(q_{i-1}, q_i)$ do not exceed $R/2^i$ and hence do not exceed $r/2^{i+1}$. Therefore the sequences $\{p_i\}$ and $\{q_i\}$ are Cauchy, converging respectively to points p_∞ and q_∞ within distance R of p_0 and q_0. Since the distance function between $\alpha(p, q_i)$ and $\alpha(p, q_\infty)$ is convex, $\{\alpha(p, q_i)\}$ converges uniformly to $\alpha(p, q_\infty)$. Similarly, $\{\alpha(p_i, q)\}$ converges to $\alpha(p_\infty, q)$. Each of these limit geodesics contains a reparametrization of $\alpha(p_\infty, q_\infty)$, so they combine to give the desired geodesic joining p and q.

We conclude, in particular, that the endpoint map sends the ball of radius $r/2$ about a geodesic γ in $\mathbf{G}_m$ isometrically onto the ball of the same radius about the endpoint of γ. $\square$

Now we indicate how the above argument on conjugate points fits into the Alexandrov theory of spaces of curvature bounded above. Following [ABN], we shall say that a geodesic space has *curvature bounded above by K* if every point has a "model neighborhood" in which any two points are joined by a minimizing geodesic in the neighborhood, and any minimizing geodesic triangle in the neighborhood has perimeter less than $2\pi/\sqrt{K}$ (if $K > 0$), and angle sum at most equal to the sum for a triangle having the same sidelengths in the standard surface S_K of constant curvature K. Alexandrov proved that then each angle individually is at most equal to its comparison angle in S_K [A1]. Here the *angle* at a vertex of a given minimizing geodesic triangle is defined to be the lim sup of the corresponding comparison angles in S_K over all triangles obtained by approaching the vertex along its adjacent sides. Curvature bounded above by 0 is a stronger condition in general than local convexity [A1]; for instance, most Minkowski spaces satisfy the latter and not the former.

The Alexandrov development method then shows that minimizing geodesics in a model neighborhood are unique and vary continuously with their endpoints ([A2], p. 51-56). The main step in this method is the proof that if one forms a triangle in a model neighborhood by moving distances x and y along two minimizing geodesics from m, then the angle in the model triangle in S_K at the point corresponding to m is nondecreasing in x and y. It follows from this by a hinge argument that the distance between any two points of a triangle in a model neighborhood is no greater than the distance in S_K between the two corresponding points of the model triangle. Alexandrov further proves that in any region in which minimizing geodesics are unique and vary continuously with their endpoints, the angle comparison property for minimizing geodesic triangles holds globally as well as locally ([A2], p. 56-58). Alexandrov's development method may also be applied to an arbitrary, not necessarily minimizing, geodesic γ in $\mathbf{G}_m$ (of length less than $\pi/\sqrt{K}$ if $K > 0$), and any two geodesics σ_1 and σ_2 sufficiently close in $\mathbf{G}_m$ to γ. We outline the argument.

We may assume that σ_1 and σ_2 lie within distance $r/2$ of γ, where r is a uniform model radius for γ, and that all geodesic triangles $\Delta(t) = m\sigma_1(t)\sigma_2(t)$ consisting of subsegments of σ_1 and σ_2 and the minimizing geodesic between their righthand endpoints have perimeters less than $2\pi/\sqrt{K}$. For all t in some interval $[0, t_0]$, the sidelengths of $\Delta(t)$ satisfy the triangle inequalities, $\Delta(t)$ and its model triangle in S_K satisfy the angle comparison property, and the angle $\theta(t)$ at the point corresponding to m in the model triangle is nondecreasing in t. We claim that these properties extend to $\min\{t_0 + \varepsilon, 1\}$, for uniform ε, whenever they extend to $t_0 < 1$; and hence they extend to $t_0 = 1$. To see this, choose ε so that the restrictions of σ_1 and σ_2 to $[t_0, t_0 + \varepsilon]$ lie in a model neighborhood of $\gamma(t_0)$ and are minimizing. Construct a pentagon in S_K in the obvious way out of three model triangles corresponding to $\Delta(t_0)$, $\Delta\sigma_1(t_0)\sigma_1(u)\sigma_2(t_0)$ and $\Delta\sigma_2(t_0)\sigma_2(u)\sigma_1(u)$ respectively. By the angle comparison property, the interior angles of this pentagon at the points corresponding to $\sigma_1(t_0)$ and $\sigma_2(t_0)$ are at least π. Thus this pentagon determines a surface with boundary in S_K whose boundary is itself a minimizing geodesic triangle in the interior metric. Therefore the triangle inequalities hold for $\Delta(u)$. By straightening the two concave sides of the pentagon one increases the three convex angles and hence obtains a model triangle for $\Delta(u)$ that satisfies the angle comparison property. The same argument applied to $\Delta(u)$ and $\Delta(v)$ for $t_0 \leqslant u < v \leqslant t_0 + \varepsilon$ shows that $\theta(u) \leqslant \theta(v)$, and hence verifies the above claim. It follows by a hinge argument that $\Delta(1)$ satisfies the following *uniform distance comparison property:* for $0 \leqslant t \leqslant 1$, the distance between $\sigma_1(t)$ and $\sigma_2(t)$ is no greater than the distance in S_K between the corresponding points of the model triangle for $\Delta(1)$. In particular, the endpoint map on $\mathbf{G}_m$ is injective on a neighborhood of γ.

One may then ask whether the endpoint map is surjective, sending a neighborhood of γ onto a neighborhood of its endpoint. To answer this question fully, we indicate how to extend the proof of Theorem 2 to the case $K > 0$:

THEOREM 3. *A complete geodesic space of curvature bounded above by $K > 0$ has no conjugate points along geodesics of length less than $\pi/\sqrt{K}$.*

It is easy to give examples showing that local injectivity of the endpoint map may not imply local surjectivity beyond length $\pi/\sqrt{K}$. (However, in Riemannian manifolds without boundary, local injectivity of the exponential map implies regularity and hence local surjectivity [W].) For instance, a closed unit hemisphere in its interior metric has curvature bounded above by 1; here the nature of $\mathbf{G}_m$ changes abruptly at length π. If γ lies on the boundary circle and has length $\pi + \varepsilon$, then a small neighborhood of γ is mapped homeomorphically onto a circular segment, not onto a neighborhood of the endpoint.

Proof. Fix a geodesic γ of length less than $\pi/\sqrt{K}$. Let $r > 0$ be a uniform radius for model balls around points of γ. Let $P(L)$ be the statement:

Given ε in $(0, r)$, there is $\delta > 0$ such that for every subsegment $\bar{\gamma}$ of γ of length at most L, any two points p and q whose respective distances from the endpoints of $\bar{\gamma}$ are less than δ are joined by a geodesic $\alpha = \alpha(p, q)$ whose distance from $\bar{\gamma}$ is less than ε and whose length is at most $L + \varepsilon$.

If $\rho = \min\{r/2, \pi/4\sqrt{K}\}$, then $P(\rho)$ holds, by taking $\delta = \min\{\varepsilon/2, \rho/2\}$. This estimate uses the the fact that the distances between corresponding points of two sides of a triangle in S_K never exceed the endpoint value if both sides have length less than $\pi/2\sqrt{K}$. It remains to be shown that if $P(L)$ holds and $L < 2\pi/3\sqrt{K}$, then $P(3L/2)$ holds.

Choose $\varepsilon < \min\{r/2, \pi/6\sqrt{K}\}$. This choice of ε ensures that any two geodesics issuing from the same point and having length at most $L + \varepsilon$ and distance at most ε from a subsegment of γ will satisfy the uniform distance comparison property. In particular, the geodesics $\alpha(p, q)$ in $P(L)$ are unique. Now choose $\varepsilon' < \min\{2\pi/3\sqrt{K} - L, 2\varepsilon/3\}$. Denote by δ' the value given by applying $P(L)$ to γ, with ε' as the desired distance from subsegments of γ. Set $L' = L + \varepsilon'$ and $\lambda = \sin(\sqrt{K}L'/2)/\sin(\sqrt{K}L')$ (then $1/2 < \lambda < 1$). It is an exercise in spherical trigonometry to show that if β_1 and β_2 are two sides of a minimizing triangle in S_K and both have length less than L', then

$$d(\beta_1(1/2), \beta_2(1/2)) < \lambda d(\beta_1(1), \beta_2(1)) . \tag{1}$$

Let $\delta = (1 - \lambda)\delta'/\lambda$ (then also $\delta < \delta'$).

Suppose that $\bar{\gamma}$ is a subsegment of γ of length $\bar{L} \leqslant 3L/2$, with endpoints $\bar{p}$ and $\bar{q}$, and let p and q be points within distance δ of these endpoints. We now follow the construction of Theorem 2. Subdivide $\bar{\gamma}$ into thirds by points p_0, q_0 and take, recursively, p_i as the midpoint of $\alpha(p, q_{i-1})$ and q_i as the midpoint of $\alpha(p_{i-1}, q)$. To verify that this recursive definition is possible, apply $P(L)$ repeatedly to the subsegments $\alpha(\bar{p}, q_0)$ and $\alpha(p_0, \bar{q})$ of γ, and note that inductively $d(p_{i-1}, p_i)$ and $d(q_{i-1}, q_i)$ are less than $\lambda^i\delta$ by the uniform distance comparison property and (1), and hence $d(p_0, p_i)$ and $d(q_0, q_i)$ are less than $\lambda\delta/(1 - \lambda) = \delta'$. In particular, $\{p_i\}$ and $\{q_i\}$ are Cauchy, and converge to p_∞ and q_∞ respectively. By the uniform distance comparison property, $\{\alpha(p, q_i)\}$ converges uniformly to $\alpha(p, q_\infty)$ and $\{\alpha(p_i, q)\}$ to $\alpha(p_\infty, q)$. These two limit geodesics overlap since $\alpha(p_\infty, q_\infty)$ is unique, hence combine to give a geodesic from p to q that has distance at most $\varepsilon' < \varepsilon$ from $\bar{\gamma}$ and length at most $\bar{L} + 3\varepsilon'/2 < \bar{L} + \varepsilon$. □

3. Proof of Theorem 1

Again consider a locally convex, complete geodesic space M, and let $\mathbf{G}_m$ be the space of geodesics starting at m carrying the uniform metric $\mathbf{d}$. It follows from local convexity that a Cauchy sequence in $\mathbf{G}_m$ converges to a geodesic and hence $\mathbf{d}$ is complete. Furthermore, M has *neighborhoods of bipoint uni-*

queness, any two points of which are joined by a unique minimizing geodesic in M, varying continuously with its endpoints. Note that neighborhoods of bipoint uniqueness guarantee that each point is the center of a contractible metric ball, since a minimizing geodesic from the center of a metric ball lies in the ball if its righthand endpoint does. Thus any interior metric space with neighborhoods of bipoint uniqueness is pathconnected, locally pathconnected and locally simply connected, and covering space theory applies. We now give a proof of Theorem 1, restated as follows:

THEOREM 4. *In a locally convex, complete geodesic space, the endpoint map on the space of geodesics from any given point is the universal covering map. Thus each homotopy class of curves between two given points contains exactly one geodesic.*

If M is simply connected, it follows that the members of $\mathbf{G}_m$ are uniquely determined by and vary continuously with their righthand endpoints. (Thus M is contractible.) Furthermore, the distance function between any two geodesics in M is convex, as claimed by Theorem 1. Indeed, for this it suffices to verify the midpoint convexity property for two sides of an arbitrary geodesic triangle (see [Bu2], p. 237); by continuity, one can subdivide into arbitrarily thin triangles, for which convexity is obvious.

We use a covering lemma for interior (rather than geodesic) spaces. It will be applied to the case in which $\overline{M}$ is $\mathbf{G}_m$, and is not known to contain minimizing geodesics between pairs of its members. By saying ϕ is a *local isometry of $\overline{M}$ onto M* we mean that every point in $\overline{M}$ has a neighborhood mapped isometrically onto a neighborhood of the image point.

LEMMA 1. *Let M and $\overline{M}$ be complete interior metric spaces. If M has neighborhoods of bipoint uniqueness, then any local isometry ϕ of $\overline{M}$ onto M is a covering map.*

Proof. Choose the open metric ball $B(p, \varepsilon)$ to be a neighborhood of bipoint uniqueness in M. Since $\overline{M}$ is not necessarily geodesic, we argue as follows to show that the restriction of ϕ to $B(\bar{p}, \varepsilon)$ is injective, for $\bar{p}$ in the preimage of p. Since $\overline{M}$ is interior, any two points of $B(\bar{p}, \varepsilon)$ may be joined to $\bar{p}$ by curves $\bar{\alpha}$ and $\bar{\beta}$ which map $[0, 1]$ into $B(\bar{p}, \varepsilon)$. Their image curves, α and β, lie in $B(p, \varepsilon)$, since a local isometry preserves lengths and hence does not increase distances. There is a continuous variation of minimizing geodesics γ_t from $\alpha(t)$ to $\beta(t)$, $0 \leqslant t \leqslant 1$. Since a local isometry between complete spaces has the unique pathlifting property, one may lift $\alpha \mid [0, t]$ followed by

γ_t, for each t, to $\bar{p}$. This gives a continuous curve starting at $\bar{p}$ and lying over β, hence coinciding with $\bar{\beta}$. If α and β have the same righthand endpoint, then γ_1 is constant, hence $\bar{\alpha}$ and $\bar{\beta}$ also have the same righthand endpoint.

From here it is straightforward to check that the preimage of $B(p, \varepsilon)$ has the desired form. For instance, the fact that $B(\bar{p}_1, \varepsilon)$ and $B(\bar{p}_2, \varepsilon)$ are disjoint for distinct $\bar{p}_1$, $\bar{p}_2$ in the preimage of p has almost the same proof as above. $\square$

Proof of Theorem 4. Note that $\mathbf{G}_m$ is contractible, hence connected and simply connected. By Lemma 1, it suffices to define a complete interior metric $\mathbf{d}^*$ on $\mathbf{G}_m$, with respect to which the endpoint map is a local isometry and whose topology agrees with that of $\mathbf{d}$. (In general $\mathbf{d}$ is not interior; for example, take M to be a Euclidean circle.) Let $\mathbf{d}^*$ be the interior metric induced by $\mathbf{d}$; that is, let $\mathbf{d}^*(\gamma, \sigma)$ be the infimum of lengths of curves in $(\mathbf{G}_m, \mathbf{d})$ from γ to σ. Since these lengths are greater than or equal to $\mathbf{d}(\gamma, \sigma)$, we have $\mathbf{d} \leqslant \mathbf{d}^*$. By Theorem 2, the endpoint map is a local isometry from $(\mathbf{G}_m, \mathbf{d})$ onto M. It follows by the definition of $\mathbf{d}^*$ that every element of $\mathbf{G}_m$ has a neighborhood on which $\mathbf{d}$ and $\mathbf{d}^*$ coincide. It only remains to verify that $\mathbf{d}^*$ is complete; but since $\mathbf{d}$ is complete and $\mathbf{d} \leqslant \mathbf{d}^*$, any $\mathbf{d}^*$-Cauchy sequence converges in $\mathbf{d}$ and hence in $\mathbf{d}^*$. $\square$

4. Cohn-Vossen's theorem and spaces without conjugate points

We have seen that in complete locally convex spaces, the endpoint map on $(\mathbf{G}_m, \mathbf{d})$, which we may denote by $\exp_m$, is a covering map. Such an argument will be more difficult to make if we merely assume that our spaces have no conjugate points; in fact, we have only been successful under the additional assumption of local compactness. Recall that the Hopf-Rinow theorem is used to prove the corresponding theorem in Riemannian geometry. To follow this lead would require a very general version of the Hopf-Rinow theorem, and one which does not hinge on the infinite extendibility of geodesics. It turns out that, in locally compact spaces, one may substitute for the notion of infinite extendibility, that of extendibility to a closed interval. This version is essentially due to Cohn-Vossen [C-V]; also see [Bu3, p. 4]. (In these references, condition (i) below is not discussed explicitly, but the proof suffices for the theorem as stated here.)

THEOREM 5 [Cohn-Vossen]. *In a locally compact, interior metric space M, the following are equivalent: (i) every halfopen minimizing geodesic from a base point extends to a closed interval; (ii) every halfopen geodesic extends to a closed interval; (iii) bounded closed subsets are compact; (iv) M is complete. Any of these implies: (v) M is a geodesic space (i.e., any two points may be joined by a shortest curve).*

Now the standard proof of the Hadamard-Cartan theorem may be adapted to give:

THEOREM 6. *In a locally compact, complete geodesic space without conjugate points, each homotopy class of curves between two given points contains exactly one geodesic.*

To do this, we modify the covering lemma. Say that a space M has *neighborhoods of radial uniqueness* if every point m is the center of a metric ball B, each of whose points can be joined to m by a unique minimizing geodesic (necessarily in B) and by no other geodesic in B. The proof of the following is entirely standard.

LEMMA 2. *Let M and $\overline{M}$ be complete geodesic spaces. If M has neighborhoods of radial uniqueness, then any local isometry of $\overline{M}$ onto M is a covering map.*

Proof of Theorem 6. Since M has no conjugate points, a sufficiently small metric ball B around m is a neighborhood of radial uniqueness. This fact and local compactness imply that the minimizing geodesic to m in B varies continuously with its endpoint. Thus B is contractible and covering space theory applies to M. Now it suffices to show that the endpoint map $\exp_m$ is a covering map.

By assumption, $\exp_m$ is a local homeomorphism from $(\mathbf{G}_m, \mathbf{d})$ onto M. Now define a new metric $\mathbf{d}^*$ on $\mathbf{G}_m$ by requiring that $\exp_m$ be a local isometry of $(\mathbf{G}_m, \mathbf{d}^*)$ onto M and $(\mathbf{G}_m, \mathbf{d}^*)$ be interior. (This will agree with the metric $\mathbf{d}^*$ of the previous section if M is locally convex, but in general they will be different.) Thus we now take $\mathbf{d}^*(\gamma, \sigma)$ to be the infimum of lengths of curves in M that are the endpoint curves of curves in $(\mathbf{G}_m, \mathbf{d})$ from γ to σ. By Lemma 2, it only remains to show that $(\mathbf{G}_m, \mathbf{d}^*)$ is a complete geodesic space. Note that $(\mathbf{G}_m, \mathbf{d}^*)$ is locally compact, being locally homeomorphic to M. Choose the constant geodesic at m as a basepoint in $\mathbf{G}_m$. A geodesic starting at m in $(\mathbf{G}_m, \mathbf{d}^*)$ projects under $\exp_m$ to a geodesic starting at m in M. Since the latter can be extended to a closed interval by the completeness of M, so can the former. By Theorem 5, since $(\mathbf{G}_m, \mathbf{d}^*)$ satisfies (i), it satisfies (iv) and (v). □

Remark 1. It is easily seen that Cohn-Vossen's theorem does not extend to spaces that are not locally compact. Indeed, an ellipsoid in Hilbert space, the lengths of whose axes are strictly decreasing and bounded above zero, satisfies all but (iii) and (v). The graph of $z = \cos x \cos(1/y)$ for $-\pi/2 \leqslant x \leqslant \pi/2$ and $y > 0$, with the straight line segment from $(-\pi/2, 0, 0)$ to $(\pi/2, 0, 0)$ adjoined, satisfies all but (iii).

Remark 2. A G-space is a locally compact, complete geodesic space which has neighborhoods of bipoint uniqueness, and in which geodesics are infinitely and uniquely extendible. In [Bu2], Busemann studies the Hadamard-Cartan theorem in the setting of G-spaces satisfying a condition that he shows is equivalent to nonpositive curvature in Riemannian manifolds but weaker in G-spaces, and which for brevity we call *local peaklessness* [Bu2, p. 269]. This means that the space is covered by neighborhoods U such that $d(\gamma(t), \sigma)$ is a peakless function for any two shortest curves γ and σ in U. Here, a *peakless* function is one whose values on any interval do not exceed the larger of the two endpoint values, with equality occurring only if the function is constant on the interval. In G-spaces, local peaklessness is equivalent to having *convex capsules* [Bu2, p. 244]. Thus Busemann's theorem is: *A simply connected G-space with convex capsules and domain invariance contains a unique geodesic joining any two of its points.* Our proof of Theorem 2 does not carry over when local convexity is replaced by local peaklessness. However, we have found a different proof that Theorem 2 holds when local convexity is replaced by local compactness and local peaklessness. This fact and Theorem 6 imply, in particular, that the theorem of Busemann just stated holds without the hypothesis of domain invariance.

REFERENCES

[ABB] ALEXANDER, S. B., I. D. BERG and R. L. BISHOP. Geometric curvature bounds in Riemannian manifolds with boundary. Preprint.

[A1] ALEXANDROV, A. D. A theorem on triangles in a metric space and some of its applications. *Trudy Math. Inst. Steks. 38* (1951), 5-23 (Russian).

[A2] ——— Über eine Verallgemeinerung der Riemannschen Geometrie. *Schr. Forschungsinst. Math. 1* (1957), 33-84.

[ABN] ALEXANDROV, A. D., V. N. BERESTOVSKII and I. G. NIKOLAEV. Generalized Riemannian spaces. *Russian Math. Surveys 41* (1986), 1-54.

[Ba] BALLMANN, W. Singular spaces of non-positive curvature. In E. Ghys, P. de la Harpe (eds.). *Sur les Groupes Hyperboliques d'après Gromov*. Birkhäuser, Boston, Basel, Stuttgart, 1990.

[BGS] BALLMANN, W., M. GROMOV and V. SCHROEDER. *Manifolds of Nonpositive Curvature*. Birkhäuser. Boston, Basel, Stuttgart, 1985.

[Bu1] BUSEMANN, H. Spaces with non-positive curvature. *Acta Mathematica 80* (1948), 259-310.

[Bu2] ——— *The Geometry of Geodesics*. Academic Press, New York, San Francisco, London, 1955.

[Bu3] ——— *Recent Synthetic Differential Geometry*. Springer, Berlin, Heidelberg, New York, 1970.

[C-V] COHN-VOSSEN, S. Existenz Kurzester Wege. *Doklady SSSR 8* (1935), 339-342.

[Gv1] GROMOV, M. Hyperbolic manifolds, groups and actions. I. Kra, B. Maskit (eds.), *Riemann Surfaces and Related Topics,* Proceedings, Stony Brook 1978, Annals of Math. Studies, Number 97, Princeton University (1981), 183-213.

[Gv2] ——— Hyperbolic groups. S. M. Gersten (ed.), *Essays in Group Theory*. Math. Sciences Research Institute Publications, Number 8, Springer-Verlag, New York, Berlin, Heidelberg (1987), 75-264.

[GLP] ——— *Structures métriques pour les variétés riemanniennes*. Rédigé par J. Lafontaine et P. Pansu, CEDIC/Fernand Nathan, Paris, 1981.

[Gn] GROSSMAN, N. Hilbert manifolds without epiconjugate points. *Proc. Amer. Math. Soc. 16* (1965), 1365-1371.

[R] RINOW, W. Die innere Geometrie der metrischen Raume. Springer, Berlin, Heidelberg, New York, 1961.

[W] WARNER, F. W. The conjugate locus of a Riemannian manifold. *Amer. J. Math. 87* (1965), 575-604.

(Reçu le 10 février 1990)

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