§1. General remarks on $I_0(M,L)$

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where $\equiv$ indicates that there is an orientation preserving diffeomorphism of pairs which is concordant to the identity map as a diffeomorphism of the ambient space M.

Our results suggest that $I(M, L)$ and $I_0(M, L)$ depend only on the order of a meridian of L in $\pi_1(M-L)$ or $H_1(M-L; \mathbf{Z})$. Roughly speaking, according as the order is infinite, 1, or p $(1<p<\infty)$, they can be distinguished by (at least) these three types:

Type 1 $I(M, L) = \{0\mathrm{N}\}$,

Type 2 $I(M, L) = \mathscr{K}_n$, $I_0(M, L) = \ker \sigma$,

Type 3 $\{0\} \subsetneqq I(M, L) \subsetneqq \mathscr{K}_n$, $\{0\} \subsetneqq I_0(M, L) \subsetneqq \ker \sigma$,

(see section 4 for $\sigma(S^{n+2}, K)$).

We refer the reader to 1.1, 2.6, 3.4, 5.1, 5.2, and 5.8 for the precise statement.

This paper consists of five sections. In Section 1, we deduce a necessary condition for $I_0(M, L)$, which is valid for any (M, L). We treat type 1 in Section 2. Type 2 is discussed in Sections 3, 4 and type 3 is discussed in Section 5. We will find that type 3 is closely related to the generalized Smith conjecture.

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§ 1. General remarks on $I_0(M, L)$

It is known (and it is easily verified) that the signature of a Seifert surface of an oriented n-knot K in S^{n+2} is independent of the choice of a Seifert surface; so it is an invariant of the oriented knot K. The invariant is called the signature of the knot K and denoted by $\mathrm{Sign}(S^{n+2}, K)$. We note that $\mathrm{Sign}(S^{n+2}, K)$ is trivially zero unless $n+1 \equiv 0$ (4).

As is seen in Section 3, there is a pair (M^{n+2}, L^n) such that $I(M, L) = \mathscr{K}_n$ for any $n \geqslant 3$. In contrast, we can deduce a necessary condition for $I_0(M, L)$ which holds for any pair (M, L).

THEOREM 1.1. *If* $(S^{n+2}, K) \in I_0(M, L)$, *then* $\mathrm{Sign}(S^{n+2}, K) = 0$.

Proof. Let V be a Seifert surface of K. Since $S^{n+2} = \partial D^{n+3}$, we can push the interior of V into the interior of D^{n+3} so that V is transverse to S^{n+2}. This yields an oriented pair (D^{n+3}, V) having (S^{n+2}, K) as the boundary.

The boundary connected sum $(M, L) \times I \natural (D^{n+3}, V)$ gives a cobordism between $(M, L) \# (S^{n+2}, K)$ and (M, L). We note that the ambient space of the cobordism is diffeomorphic to $M \times I$. Since $(S^{n+2}, K) \in I_0(M, L)$, there is an orientation preserving diffeomorphism $f: (M, L) \# (S^{n+2}, K) \to (M, L)$ which is concordant to the identity when regarded as a diffeomorphism of the ambient space M. We paste togethor $(M, L) \# (S^{n+2}, K)$ and (M, L) by f to get an oriented pair of closed manifolds. Since f is concordant to the identity, the resulting ambient space is diffeomorphic to $M \times S^1$. We shall denote by X the resulting oriented closed submanifold of $M \times S^1$.

The additivity property of the signature (see [AS, p. 588]) says that

$$\operatorname{Sign} X = \operatorname{Sign} L \times I + \operatorname{Sign} V = \operatorname{Sign} V,$$

where $\operatorname{Sign} L \times I = 0$ follows easily from the definition of the signature of a manifold with boundary. By the Hirzebruch signature theorem (see [MS, § 19]) we have

$$\operatorname{Sign} X = \mathscr{L}(X)[X]$$

where the right hand side means the Hirzebruch L-class $\mathscr{L}(X)$ of X evaluated on the fundamental class $[X]$ of X. In the sequel we shall show $\mathscr{L}(X)[X] = 0$.

Let $j: X \to M \times S^1$ be the inclusion map. Then it is not difficult to see that

(1.2) $$j_*[X] = [L \times S^1] \quad \text{in} \quad H_{n+1}(M \times S^1; \mathbf{Z})$$

where $[L \times S^1]$ denotes the homology class represented by $L \times S^1$.

Let ν be the normal bundle to X in $M \times S^1$. By the multiplicativity of L-class we have

(1.3) $$\mathscr{L}(X) = \mathscr{L}(\nu)^{-1} j^* \mathscr{L}(M \times S^1)$$
$$\mathscr{L}(M \times S^1) = \mathscr{L}(M) \times \mathscr{L}(S^1) = \pi^* \mathscr{L}(M)$$

where $\pi: M \times S^1 \to M$ is the projection map. Since $\dim \nu = 2$, we have

(1.4) $$\mathscr{L}(\nu) = 1 + p_1(\nu)/3 = 1 + e(\nu)^2/3$$

where p_1 and e denote the first Pontrjagin class and the Euler class respectively.

On the other hand it is known that

(1.5) $$e(\nu) = j^* j_! (1)$$

where $j_! : H^q(X;\mathbf{Z}) \to H^{q+2}(M \times S^1;\mathbf{Z})$ denotes the Gysin homomorphism and $1 \in H^0(X;\mathbf{Z})$ is the unit element. Remember the definition of $j_!$. It is defined so that the following diagram commutes:

$$\begin{array}{ccc} H^q(X;\mathbf{Z}) & \xrightarrow{j_!} & H^{q+2}(M \times S^1;\mathbf{Z}) \\ \downarrow \cap[X] & & \downarrow \cap[M\times S^1] \\ H_{n+1-q}(X;\mathbf{Z}) & \xrightarrow{j_*} & H_{n+1-q}(M \times S^1;\mathbf{Z}) \end{array}$$

where the vertical maps are the Poincaré dualities. It says that

$$j_!(1) \cap [M \times S^1] = j_*[X].$$

This together with (1.2) means that

$$j_!(1) \in \pi^* H^2(M;\mathbf{Z}).$$

Hence it follows from (1.4) and (1.5) that

$$\mathscr{L}(\nu) \in j^*\pi^*H^*(M;Q)$$

and hence

$$\mathscr{L}(X) \in j^*\pi^*H^*(M;Q)$$

by (1.3). This together with (1.2) implies that

$$\mathscr{L}(X)[X] = 0. \qquad \text{Q.E.D.}$$

Theorem 1.1 gives a necessary condition for (S^{n+2}, K) to belong to $I_0(M, L)$. When we consider the converse problem, i.e. the problem to find (S^{n+2}, K) in $I_0(M, L)$, we apply the relative s-cobordism theorem. We shall state it as a lemma for later convenience's sake.

LEMMA 1.6. *Suppose there exists a cobordism* (U, Z) *between* $(M, L) \# (S^{n+2}, K)$ *and* (M, L) *such that*

(1) Z *is diffeomorphic to* $L \times I$,

(2) *the exterior* $E(Z)$ *of* Z *is an s-cobordism relative boundary.*

Then $(S^{n+2}, K) \in I_0(M, L)$.

Proof. The relative s-cobordism theorem says that $E(Z)$ is diffeomorphic to $E(L) \times I$ where the diffeomorphism can be taken as the identity on $E(L) \times \{0\}$ and $(\partial E(L)) \times I$. Therefore it extends to a diffeomorphism: $(U, Z) \to (M, L) \times I$ which is the identity on the 0-level. This means that $(S^{n+2}, K) \in I_0(M, L)$. Q.E.D.