2. IRREGULAR STATES

2. Irregular states

Although introduced already in the leaflet [8] to the original **TH** puzzle, Lucas's second problem (cp. also [9]) has not yet received an adequate mathematical treatment. The reason is that the violation of the regularity assumption on the initial state takes away a great deal of symmetry from the considerations. In particular, the mathematical model has to be changed.

2.0. Mathematical model

With pegs 0, 1, 2 set up from left to right and counting positions of discs and bottoms of pegs from top left to bottom right in a given state, one can attach to each *disc* $d \in \{1, ..., n\}$ its position $\rho(d)$ in this enumeration, and to the *bottom of peg* $i \in \{0, 1, 2\}$ its position $\rho(n+1+i)$. This leads to the following definition.

Definition 11.

$$\mathfrak{T}_n := \{\rho : \{1, ..., n+3\} \underset{\text{onto}}{\overset{1-1}{\rightarrow}} \{1, ..., n+3\} ; \rho(n+1) < \rho(n+2) < \rho(n+3) \doteq n+3\} .$$

As any $\rho \in \mathfrak{T}_n$ corresponds to a state of the **TH**, it follows immediately:

Theorem 5. *The number of states of the* **TH** *with* n *discs is* $\frac{(n+2)!}{2}$.

Remark 7. Surprisingly, Lucas writes that for $n = 64$, this number has "more than fifty figures" (see [8]); although this is true, it falls short by some fourty powers of ten!

While the description of a state is simple, the rules of a move are clumsy in this model and far from intuition. So it is convenient to construct the following imbedding.

Definition 12.

$$J : \mathfrak{T}_n \overset{1-1}{\rightarrow} \{(r, h); r : \{1, ..., n+3\} \rightarrow \{0, 1, 2\}, h : \{1, ..., n+3\} \rightarrow \{0, ..., n\}\} ,$$

$$\rho \mapsto (r, h) ,$$

where

$$\forall d \in \{1, ..., n+3\} :$$
$$h(d) = \min\{\rho(n+1+i) - \rho(d); \rho(n+1+i) \geqslant \rho(d), i \in \{0, 1, 2\}\} ,$$

and $r(d)$ is the i for which this minimum is attained.

It is easily checked that J is an injection, and so $\mathfrak{T}_n$ and $J\mathfrak{T}_n$ will be identified, i.e. ρ and (r, h) will be used interchangeably. Furthermore, as $r(d)$ and $h(d)$ do not depend on ρ for $d = n + 1 + i$, $i \in \{0, 1, 2\}$, r will be identified with $r \mid \{1, ..., n\} \in T_n$ and h with $h \mid \{1, ..., n\}$. $\rho \in \mathfrak{T}_n$ will also be written $[(r(1), h(1)), ..., (r(n), h(n))]$. Again, $r(d)$ is the peg onto which disc d is stacked and $h(d)$ is its level above the bottom of that peg. In addition, by $r \mapsto (r, h)$ with $\forall d \in \{1, ..., n\}: h(d) = |\{c \in \{d, ..., n\}; r(c) = r(d)\}|$ an injection is given from T_n into $\mathfrak{T}_n$ and again r and (r, h) will be identified.

Definition 13. A pair $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$ is a *(legal) move* (of disc d from peg i to peg j), iff

$$\exists (i, j) \in \{0, 1, 2\}^2, i \neq j : d := \operatorname{top}(\rho_0; i) < \min\{n+1, \operatorname{top}(\rho_0; j)\}$$
$$\wedge \left(r_1(d) = j, h_1(d) = h_0(\operatorname{top}(\rho_0; j)) + 1, \forall c \in \{1, ..., n\} \setminus \{d\}: r_1(c) = r_0(c), \; h_1(c) = h_0(c)\right),$$

where

$$\forall \rho \in \mathfrak{T}_n \; \forall i \in \{0, 1, 2\}: \operatorname{top}(\rho; i) \in \{1, ..., n+3\}$$

with

$$r(\operatorname{top}(\rho; i)) = i, h(\operatorname{top}(\rho; i)) = \max h(r^{-1}(\{i\})).$$

For $(\sigma, \tau) \in \mathfrak{T}_n^2$, a *path* $\pi \in \Pi_n(\sigma, \tau)$ from σ to τ and its *length* are defined as in Chapter 1.

Remark 8. If ρ_0 is regular in a move $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$, then so is ρ_1, and (ρ_0, ρ_1) is a legal move in the sense of Definition 1. As the same applies to paths, it is clear that no new paths between regular states turn up.

The analogue to Definition 2i is

Definition 14. For any $\rho \in \mathfrak{T}_n$ and $d \in \{1, ..., n+3\}$ let

$$U_\rho^d := \{c \in \{1, ..., n\}; r(c) = r(d) \wedge h(c) \leqslant h(d)\}$$

and define $\bar{\rho}^d \in \mathfrak{T}_{n-h(d)}$ by

$$\forall c \in \{1, ..., n-h(d)\}: \bar{r}^d(c) = r(\iota(c)),$$
$$\bar{h}^d(c) = \begin{cases} h(\iota(c)) - h(d), & \text{if} \quad r(\iota(c)) = r(d), \\ h(\iota(c)), & \text{else}, \end{cases}$$

where $\iota: \{1, ..., n-h(d)\} \to \complement U_\rho^d$ is strictly increasing; if $d = n$, $\bar{\rho}^d$ will be written $\bar{\rho}$ simply.

Similarly, $\underline{\rho}_d \in \mathfrak{T}_{h(d)}$ is defined by

$$\forall c \in \{1, ..., h(d)\} : \underline{r}_d(c) = r(d), \underline{h}_d(c) = h(\iota(c)),$$

where now $\iota: \{1, \ldots, h(d)\} \to U_\rho^d$ is strictly increasing.

Remark 9. Given U_ρ^d, it is possible to reconstruct ρ from $\underline{\rho}_d$ and $\bar{\rho}^d$. Thus, as long as disc d does not move, any move $(\bar{\rho}_0, \bar{\rho}_1) \in \mathfrak{T}_{n-h(d)}^2$ is equivalent to a move $(\rho_0, \rho_1) \in \mathfrak{T}_n^2$, provided that $d > \max \complement U_\rho^d$. This will frequently be used in the sequel.

2.1. Existence of a shortest path from a state to a regular state and an upper bound for its length

In contrast to the situation for regular states, $(\rho_1, \rho_0) \in \mathfrak{T}_n^2$ is not necessarily a legal move if (ρ_0, ρ_1) is. So one can not expect $\Pi_n(\sigma, \tau)$ to be non-empty for every pair $(\sigma, \tau) \in \mathfrak{T}_n^2$. The goal of this section will be:

THEOREM 6. *Let $n \in \mathbf{N}\backslash\{1\}$. For any pair $(\sigma, t) \in \mathfrak{T}_n \times T_n$ there is a (shortest) path from σ to t with length less than or equal to $2^n - 1 + 2^{n-2}$.*

Remark 10. i) The restriction on n is not serious, since there are no irregular states for $n \in \{0, 1\}$.

ii) The bound on the length of a shortest path in Theorem 6 is sharp:

Example 2. $\sigma = [(0, n), (0, n-1), ..., (0, 3), (0, 1), (0, 2)]$, $t = \hat{0}^n$. Before the first move of disc n (it has to be moved to arrive at a regular state!), to peg 1 for instance, discs 1 to $n - 2$, which are regularly distributed on top of it, have to be moved to peg 2. So, by Theorem 2, at least 2^{n-2} moves have been carried out after the first move of disc n, when a regular state is reached from which it takes another $2^n - 1$ moves to arrive at t, as can be calculated using Theorem 3.

To prove Theorem 6, some preparations have to be done.

LEMMA 3. *For every $\rho \in \mathfrak{T}_n$ and $j \in \{0, 1, 2\}$ there is a $\tilde{\rho} \in \mathfrak{T}_n$ with $\tilde{r} = \hat{j}^n$ and a path from ρ to $\tilde{\rho}$ with length less than or equal to $2^n - 1$; if $n = 0$ or $r(n) \neq j$, then $\tilde{\rho}$ is regular.*

Proof by induction on n. a) The case $n = 0$ is trivial.

b) If $r(n+1) = j$, then the induction hypothesis can be applied to $\bar{\rho}$, resulting in a $\tilde{\rho}$ and a path from ρ to $\tilde{\rho}$ in the spirit of Remark 9.

If $r(n+1) \neq j$, then transfer $\bar{\rho}$ to $j \circ r(n+1)$, which takes at most $2^n - 1$ moves by hypothesis, move disc $n+1$ to j and then the first n discs to j, together at most $2^{n+1} - 1$ moves. As in the last action (if $n \neq 0$) disc n started from a peg different from j, the resulting state is regular by hypothesis. □

This lemma leads to the following interesting result:

PROPOSITION 8. *Let* $n \in \mathbf{N} \setminus \{1\}$. *For any* $\sigma \in \mathfrak{T}_n$ *there is a* $\tilde{t} \in T_n$ *and a path from* σ *to* $\tilde{t}$ *with length less than or equal to* 2^{n-2}.

Remark 11. Here again Example 2 shows that the bound on the length is sharp: Suppose for the σ of Example 2 there is a $\tilde{t} \in T_n$ and a path from σ to $\tilde{t}$ of length less than 2^{n-2}; then by Theorem 1, there is a path from σ to $t = \hat{0}$ of length less than $2^n - 1 + 2^{n-2}$, which contradicts the discussion of Example 2.

Proof of Proposition 8 by induction on n. a) For $n = 2$, the only irregular states are $[(j, 1), (j, 2)]$ for $j \in \{0, 1, 2\}$. Here it suffices to move disc 2 to a different peg to get a regular state.

b) If $h(n+1) = 1$, then the induction hypothesis can be applied to $\bar{\sigma}$. Otherwise, the transfer of $\bar{\sigma}$ to a peg j different from $s(n+1)$ and $s(n+1-h(n+1))$ is achieved in at most $2^{n+1-h(n+1)} - 1$ moves by Lemma 3. Then move disc $n+1$ to $j \circ s(n+1)$. If $h(n+1) = 2$, the resulting state is regular and the number of moves at most 2^{n-1}. Otherwise, discs 1 to n can be transferred to a regular state in at most 2^{n-2} moves by hypothesis, and the total number of moves is less than or equal to $2^{n+1-h(n+1)} + 2^{n-2} \leqslant 2^{n-1}$. □

Now the proof of Theorem 6 is a trivial combination of Proposition 8 and Theorem 1.

Although Example 2 shows that shortest paths may be as long as $2^n - 1 + 2^{n-2}$, this worst case will not occur very frequently, as the following proposition tells, which will also be important in the subsequent sections.

PROPOSITION 9. *Let* $n \in \mathbf{N}\setminus\{1\}, (\sigma, t) \in \mathfrak{T}_n \times T_n$. *Then*

$$\mu(\sigma, t) \geqslant 2^n \Rightarrow (s(n) = t(n) \wedge h(n) > 1) .$$

Proof. The proof is by constructing paths from σ to t shorter than 2^n for all cases different from the r.h.s. For convenience suppose that $n \geqslant 3$ (for $n = 2$, cp. the proof of Proposition 8).

i) $s(n) = t(n) \wedge h(n) = 1$. Then bring $\bar{\sigma}$ to $\bar{t}$, which takes at most $2^{n-1} - 1 + 2^{n-3}$ moves by Theorem 6.

ii) $s(n) \neq t(n) \wedge (h(n) > 1 \vee s(n-1) = s(n) \circ t(n))$. Then bring $\bar{\sigma}$ to peg $s(n) \circ t(n)$ in at most $2^{n-2} - 1$ moves (by Lemma 3), move disc n to $t(n)$ and then the other discs to $\bar{t}$ in at most another $2^{n-1} - 1 + 2^{n-3}$ moves by Theorem 6.

iii) $s(n) \neq t(n) \wedge (h(n) = 1 \wedge s(n-1) \neq s(n) \circ t(n))$. Then move $\bar{\sigma}$ to $\widehat{s(n) \circ t(n)}^n$ in at most $2^{n-1} - 1$ moves (by Lemma 3), move disc n to $t(n)$ and finally $\widehat{s(n) \circ t(n)}^n$ to $\bar{t}$ in at most $2^{n-1} - 1$ moves by Theorem 1. □

Remark 12. As in Theorem 1, the proof of Theorem 6 (Proposition 9) is constructive, allowing (if $s(n) \neq t(n) \vee h(n) = 1$) to find a path from σ to t with at most $2^n - 1 + 2^{n-2}$ $(2^n - 1)$ moves. But again, it does not necessarily lead to a shortest path, even if the steps are carried out efficiently; see Example 3 below. So the construction of shortest paths has to be discussed further.

2.2. CONSTRUCTION OF SHORTEST PATHS FROM A STATE TO A REGULAR STATE

Although it is now possible, in principle, to find all shortest paths from a state σ to a regular state t by sheer listing the paths between them not longer than the upper bound in Theorem 6, this crude proceeding is neither efficient nor does it provide any a priori information about the number of shortest paths. The following three lemmas will help to overcome these weaknesses.

LEMMA 4. *Let* $\pi \in \Pi_{n+1}(\sigma, t)$ *be shortest. Then disc* $n + 1$ *does not move twice to the same peg; consequently, it moves at most three times.*

Proof. Suppose $j \in \{0, 1, 2\}$ appears as goal of disc $n + 1$ at least twice in π, in moves μ' and $\mu''(\mu' < \mu'')$ say. Then, as $h_\mu(n+1) = 1$ after the first move of $n+1$, one can leave out all the moves μ with $d_\mu(\pi) = n+1$ and $\mu' < \mu \leqslant \mu''$ and gets a shorter path from σ to t. $\square$

LEMMA 5. *Let* $j \in \{0, 1, 2\}, (\sigma, \tau) \in \mathfrak{T}_n^2$ *with* $t = \hat{j}^n$. *Then*

$$\Pi_n(\sigma, \tau) \neq \emptyset \Leftrightarrow \exists d \in \{1, ..., n+3\}, h_\sigma(d) = n$$
$$\vee\ d > \max \complement\, U_\sigma^d : U_\tau^d = U_\sigma^d \wedge \underline{\tau_d} = \underline{\sigma_d} \wedge \bar{\tau}^d = \hat{j}^{n-h_\sigma(d)}.$$

Proof. "$\Rightarrow$": If τ is regular, then take $d > n$. Otherwise

$$\{d \in \{1, ..., n\}; \exists c \in \{1, ..., d-1\} : h_\tau(c) = h_\tau(d)-1\} \neq \emptyset.$$

Choose the d with $h_\tau(d)$ a maximum. Then $h_\tau(d) = n$ or $d > \max \complement\, U_\tau^d$, and $\bar{\tau}^d = \hat{j}^{n-h_\tau(d)}$. Furthermore, as there is a path from σ to τ, $U_\sigma^d = U_\tau^d$ and $\underline{\sigma_d} = \underline{\tau_d}$, and so also $h_\sigma(d) = h_\tau(d)$.

"$\Leftarrow$" follows from Theorem 6. If τ is not regular, a path from σ to τ is given by a path from $\bar{\sigma}^d$ to $\bar{\tau}^d$ fixing disc d and the discs under it. $\square$

LEMMA 6. *Let* $j \in \{0, 1, 2\}$.

i) *Let* $\sigma \in \mathfrak{T}_n, \tau_1, \tau_2$ *as in Lemma 5 for* d_1, d_2 *with* $h(d_1) \geqslant h(d_2)$. *Then* $\mu(\sigma, \tau_1) \leqslant \mu(\sigma, \tau_2)$.

ii) *Let* $i, k \in \{0, 1, 2\}, i \neq k, (\sigma_1, \sigma_2) \in \mathfrak{T}_n^2$ *with*

$$\forall d \in \{1, ..., n\} : (s_1(d) = s_2(d) =: s(d) \wedge s(d) \neq k \wedge (s(d) = i \Rightarrow h_1(d) = h_2(d))$$
$$\wedge\ (s(d) = i \circ k \Rightarrow h_2(d) = |\{c \in \{d, ..., n\}; s(c) = i \circ k\}|));$$

let $\tau_\kappa (\kappa \in \{1, 2\})$ *be as in Lemma 5 applied to* σ_κ *and* d_κ *with* $h_\kappa(d_\kappa)$ *a maximum.*

Then $\mu(\sigma_1, \tau_1) \leqslant \mu(\sigma_2, \tau_2)$.

Proof. i) Take a shortest path from σ to τ_2 and skip the moves of discs in $U_\sigma^{d_1}$.

ii) By induction on n. a) The case $n = 0$ is trivial.

b) By part i, it suffices to construct a path π_1 from σ_1 to peg j not longer than π_2, a given shortest path from σ_2 to τ_2.

The first, and possibly only, part of π_2 is equivalent to a path from $\bar{\sigma}_2$ to some peg $\tilde{j} \in \{0, 1, 2\}$. Define $\tilde{\sigma}_1 \in \mathfrak{T}_{n+1-h_2(n+1)}$ by deleting discs in $U_{\sigma_2}^{n+1}$ from σ_1 analogously to Definition 14. Then, by induction, there is a path from $\tilde{\sigma}_1$ to peg $\tilde{j}$ not longer than the former and by deleting all the moves of discs in $U_{\sigma_1}^{n+1}$ one gets a path $\tilde{\pi}_1$ from $\bar{\sigma}_1$ to peg $\tilde{j}$. If $s(n+1) = j$, then disc $n+1$ does not move in π_2, whence $\tilde{j} = j$ and

$\pi_1 = \tilde{\pi}_1$ does the job. Otherwise, add to $\tilde{\pi}_1$ the move, also present in π_2, of disc $n+1$ from $s(n+1)$ to $\tilde{j} \circ s(n+1)$. Now, if $s(n+1) = i \circ k$, a perfect n-state (perfect substate if $\tilde{j} = i$) moves from $\tilde{j}$ to some other peg in π_2, while in π_1 the latter peg can be reached in at most as many moves by Theorem 2 or Proposition 9. After that, or if $s(n+1) = i$, the induction hypothesis provides the rest of path π_1. $\square$

By Lemma 4, the possible patterns of movements of the largest disc $n+1$ in a shortest path are determined, while Lemma 5 limits the number of cases to be considered before each move of disc $n+1$. After the last move of disc $n+1$, the other discs have to be brought to $\bar{t}$. This leads to a recursive construction of all shortest paths from σ to t. Lemma 6, finally, makes this construction more efficient by pointing out the advantages of leaving the intermediate states as irregular as possible.

While Example 1 revealed that even in the case of a regular initial state uniqueness of the shortest path does not hold and that there are shortest paths with two moves of the largest disc, the following example indicates that things are even more complicated now.

Example 3. $\sigma = [(2, 1), (2, 2), (0, 1), (2, 3), (0, 2)]$, $t = [1, 1, 1, 1, 2]$. Then a careful analysis shows that a path from σ to t needs at least 11 moves if disc 5 moves only once and 22 if it moves exactly twice, but there is a shortest path of length 9 where disc 5 moves three times! As in the construction of Theorem 6 (Proposition 9) disc 5 would not move but once, this example also verifies the assertions in Remark 12.

This shows that in general the number of candidates for a shortest path may still be considerable. That is not so if t is perfect. So the rest of this chapter is devoted to the final analysis of Lucas's second problem.

2.3. Uniqueness of the solution to Lucas's second problem

The goal of this section is the following satisfying result.

THEOREM 7. *Let* $\rho \in \mathfrak{T}_n$ *and* $j \in \{0, 1, 2\}$. *Then the shortest path from* ρ *to* $\hat{j}$ *is unique, except for the case* $r = \hat{j} \wedge \rho \neq \hat{j}$, *when there are exactly two shortest paths, generated from each other by interchanging the roles of the elements of* $\{0, 1, 2\} \backslash \{j\}$.

As in the case of regular states, it will be important to know how often the largest disc will be moved in a shortest path.

LEMMA 7. *Let* $n \in \mathbf{N}, j \in \{0, 1, 2\}, \pi \in \Pi_{n+1}(\rho, \hat{j})$ *be shortest. Then disc* $n+1$ *moves*

o) *not at all if* $r(n+1) = j$ *and* $h(n+1) = 1$,

i) *exactly once if* $r(n+1) \neq j$,

ii) *exactly twice if* $r(n+1) = j$ *and* $h(n+1) > 1$.

Proof. o) If there are moves of disc $n+1$ in π, delete them all to arrive at a strictly shorter $\tilde{\pi} \in \Pi_{n+1}(\rho, \hat{j})$.

i) The possibilities of two or three moves of the largest disc $n+1$ in a shortest path π will be excluded by constructing a strictly shorter path $\tilde{\pi}$ with only one move of disc $n+1$.

Suppose disc $n+1$ moves three times. Then, by Lemma 4, its sequence of moves is necessarily from $r(n+1)$ through $j \circ r(n+1)$ and again $r(n+1)$ to j. Also, if μ is the number of the last move of disc $n+1$, π_μ is regular with $p_\mu(n+1) = j$ and $\bar{p}_\mu = \widehat{j \circ r(n+1)}^{\,n}$ and thus, by Theorem 2, $\mu_\pi = \mu + 2^n - 1$. Now carrying out the first $\mu - 1$ moves of π, skip every move of discs in U_ρ^{n+1}, then move disc $n+1$ to j. This gives a path from ρ to $\tilde{\pi}_{\tilde{\mu}}$ with $\tilde{p}_{\tilde{\mu}}(n+1) = j$, $\bar{\tilde{h}}_{\tilde{\mu}}(n+1) = 1$ and $\bar{\tilde{p}}_{\tilde{\mu}}(n) \neq j$, so that, by Proposition 9, $\hat{j}^{\,n+1}$ is reached in at most another $2^n - 1$ moves, resulting, as $\tilde{\mu} < \mu$ by at least two moves of disc $n+1$, in a path from ρ to $\hat{j}$ shorter than π.

Suppose disc $n+1$ moves twice. Then these moves, with numbers μ' and μ'' say, are necessarily from $r(n+1)$ through $j \circ r(n+1)$ to j. Carrying out only those of the first μ' moves of π with discs in $\complement\, U_\rho^{n+1}$, one arrives at a $\tilde{\pi}_{\tilde{\mu}'}$ with $\bar{\tilde{p}}_{\tilde{\mu}'} = \hat{j}^{\,n+1-h(n+1)}$. Leaving disc $n+1$ at $r(n+1)$, one proceeds by carrying through those moves μ of π with $\mu' < \mu < \mu''$ and $d_\mu(\pi) \in \complement\, U_\rho^{n+1}$, but changing the roles of $r(n+1)$ and $j \circ r(n+1)$ for $i_\mu(\pi)$ and $j_\mu(\pi)$. One arrives at $\tilde{\pi}_{\tilde{\mu}''-1}$ with $\bar{\tilde{p}}_{\tilde{\mu}''-1} = \widehat{j \circ r(n+1)}^{\,n+1-h(n+1)}$ and $\underline{\tilde{p}_{\tilde{\mu}''-1}} = \underline{\rho}$, allowing disc $n+1$ to be moved to j. Now, by Lemma 5 applied to $\sigma = \bar{\pi}_{\mu'}$ and $\tau = \bar{\pi}_{\mu''}$, $\bar{\pi}_{\mu''}$ is either regular on $r(n+1)$, in which case, by Proposition 9, $\mu(\tilde{\pi}_{\tilde{\mu}''}, \hat{j}) \leqslant 2^n - 1 = \mu(\pi_{\mu''}, \hat{j})$, or

$$\exists d \in \{1, \dots, n\}, h_\sigma(d) = n \vee d > \max \complement\, U_\sigma^d : U_\tau^d = U_\sigma^d \wedge \underline{\tau_d} = \underline{\sigma_d} \wedge$$
$$\bar{\tau}^d = \widehat{r(n+1)}^{\,n-h_\sigma(d)};$$

but then discs in U_ρ^d have not been moved neither in the first μ'' moves of π nor in the first $\tilde{\mu}''$ moves of $\tilde{\pi}$. Let μ''' be the first move of d in π, so that $\mu''' = \mu'' + 2^{n-h_\rho(d)}$; on the other hand, state $\pi_{\mu'''}$ can be reached from $\tilde{\pi}_{\tilde{\mu}''}$ in at most $2^{n-h_\rho(d)}$ moves by Proposition 9, since for

$$\tilde{d} := \max \complement U^{d}_{\bar{\tilde{\pi}}_{\tilde{\mu}''}} : \bar{\tilde{p}}_{\tilde{\mu}''}(\tilde{d}) = r(n+1) \vee \bar{\tilde{h}}_{\tilde{\mu}''}(\tilde{d}) = 1 .$$

ii) Disc $n+1$ has to be moved at least once. After its first move, situation i is reached. □

The last step shows that the only possible ambiguity in the sequence of moves of the largest disc might arise in case ii of Lemma 7 by the question to which of the pegs $\neq j$ it should be moved. Lemma 8 answers this question.

LEMMA 8. *Let* $(i, j) \in \{0, 1, 2\}^2, i \neq j, \rho \in \mathfrak{T}_n$ *with* $r(n) = i \circ j$. *Then*

$$\mu(\rho, \hat{i}) = \mu(\rho, \hat{j}) \Leftrightarrow r = \widehat{i \circ j} .$$

Proof. "$\Leftarrow$" is trivial by interchanging i and j.

"$\Rightarrow$" will be proved by induction on n.

a) Cases $n = 0$ and $n = 1$ are trivial.

b) Suppose $\{c \in \{1, ..., n\} ; r(c) \neq i \circ j\} \neq \emptyset$. Let π_i, π_j be shortest paths from ρ to $\hat{i}^{n+1}$ and $\hat{j}^{n+1}$, respectively, and let $d := \max \complement U_\rho^{n+1}$.

If $r(d) = i \circ j$, then $\mu(\rho, \hat{i}) = \mu(\bar{\rho}, \hat{j}) + 2^n$ and $\mu(\rho, \hat{j}) = \mu(\bar{\rho}, \hat{i}) + 2^n$ by Lemma 7, Lemma 3 and Theorem 2. But by induction hypothesis $\mu(\bar{\rho}, \hat{i}) \neq \mu(\bar{\rho}, \hat{j})$.

If, without loss of generality, $r(d) = i$, then in π_i leave out the first move of disc d, go on until the move of disc $n+1$ ignoring the moves of discs in U_ρ^d and interchanging i and j in the moves of the other discs; then move disc $n+1$ to j. To the rest of the moves, Lemma 6 can be applied (again interchanging i and j), yielding a path from ρ to $\hat{j}$ strictly shorter (by at least one move of disc d) than π_i. □

Now Lemmas 5 to 8 comprise all the information necessary to prove Theorem 7.

Proof of Theorem 7 by induction on n. a) Case $n = 0$ is trivial.

b) If $r(n+1) = j$ and $h(n+1) > 1$, then there are still two possible sequences of moves for disc $n+1$, differing in the intermediate peg to be passed. Let $d := \max \complement U_\rho^{n+1}$. If $r(d) = j$, then Lemma 8 can be applied. Otherwise the path which moves disc $n+1$ to $j \circ r(d)$ is strictly shorter than the one with intermediate peg $r(d)$ by an argument similar to that in the proof of Lemma 8 and with the aid of Lemma 6.

In all the other cases, the moves of disc $n + 1$ are determined by Lemma 7, the moves of the other n discs are governed by Lemmas 5 and 6, and their uniqueness follows by induction hypothesis, keeping in mind that the paths of Lemma 5 are actually paths from $\bar{\sigma}^d$ to $\hat{j}^{n-h_\sigma(d)}$. $\square$

Using the methods of this chapter, one finds the shortest path from σ to $\hat{0}$ in Figure 1 with length 102.

3. Open problems

Much of the discussion of the **TH** in computer science literature has been a controversy between recursion and iteration. It has turned out here that problems involving just regular states, can be solved by iteration very elegantly (Chapter 1). On the other hand, as soon as irregular states are considered, only recursive solutions are available (Chapter 2). While for $\mathfrak{P}3$ the solution is essentially unique and the recursion will work efficiently, the situation for $\mathfrak{P}4$ is less straightforward. Although the number of cases to be considered can be further limited by methods as in Section 2.3 (e.g. the shortest path (of length 108) from σ to r in Figure 1 is unique), and one can show that no three moves of the largest disc $n + 1$ occur if $r(n+1) = t(n+1)$ and $h(n+1) > 1$, it is not clear whether there are shortest path problems with even three different solutions. Also it seems that the minimal length in $\mathfrak{P}s$ 3 and 4 can only be determined recursively.

The only existing solution to the **TH** with more than three pegs is also recursive, and the preceeding chapters should have demonstrated that things are not as easy as many authors might hope (see the remarks in the Introduction). To move the largest disc $n + 1$ in the solution of $\mathfrak{P}0$ with four pegs, the n other discs have to be transferred to two different pegs; after the last move of disc $n + 1$, discs 1 to n have to be sent from some two pegs to the top of disc $n + 1$. Again it follows by symmetry that disc $n + 1$ will only be moved once in a shortest path. But this time, this does not reduce the problem for $n + 1$ discs to a similar one with only n discs, but to the different setting of how to transfer n discs from a perfect state to two different pegs in the shortest possible way. Here is where the hitherto unjustified assumption made in literature enters, namely that this will be achieved by dividing the perfect state in a suitable way into two parts, then first solving $\mathfrak{P}0$ for the smaller discs using four pegs, leaving them untouched thereafter and solving the old problem for the larger discs using three pegs only.