5. Kähler-Einstein Metrics on Noncompact Manifolds

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$c_1(L) \geqslant 0$ on M and $c_1(L) > 0$ outside some subvariety of M. Siu [S5] proved that the converse is also true under the weaker assumption that $c_1(L)$ is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (**) is Moishezon. It is also not known whether $\mathbf{CP}^n$, $n \geqslant 4$, can admit a nonstandard structure which is Moishezon. For $n = 3$, T. Peternell [Pe] proved that if M is a Moishezon 3-fold which is topologically isomorphic to $\mathbf{CP}^3$, then M is the standard $\mathbf{CP}^3$. His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological $\mathbf{CP}^n$.

5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let g be a complete Kähler-Einstein metric on M^n, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$ for some constant c. If $c > 0$, Myer's theorem would imply N is compact. Hence, $c \leqslant 0$ and $c_1(M) \leqslant 0$. In this section we consider the case $c_1(M) < 0$ and leave the case $c_1(M) = 0$ for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics $g_{i\bar{j}}$ with $R_{i\bar{j}} = -g_{i\bar{j}}$. In particular, one would like to impose conditions on M to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if M and N are complete Kähler-Einstein manifolds with $R = -1$ and $F\colon M \to N$ is a biholomorphism, then F is an isometry. To prove this, let g and dv and g' and dv' denote the Kähler-Einstein metrics and volume forms of M and N, respectively. If we let $\rho = \log(F^*dv'/dv)$, then $\partial\bar{\partial}\rho = -f^*\,\mathrm{Ric}' + \mathrm{Ric} = F^*g' + g$. Taking traces, we have $\Delta\rho = -n + n \cdot e^{\rho/n}$. Hence, the maximum principle implies $\rho \leqslant 0$ and $F^*dv' \leqslant dv$. Replacing F by F^{-1}, we have $F^*dv' \geqslant dv$ and F is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric ds^2 on M is said to be almost complete if we can write M as an increasing union of domains Ω_α and there exist complete metrics ds_α^2 on Ω_α for each α such that ds_α^2 converges to ds^2 on compact subsets of M. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of M. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than a negative constant on M, then the pseudomeasure in the sense of Eiseman is in fact a measure, that is to say, M is measure hyperbolic.

In [C-Y1], Cheng and Yau obtained the existence of Kähler-Einstein metrics on a large class of noncompact manifolds. More precisely, they proved the following. Let M^n be a Hermitian manifold whose Ricci tensor defines a Kähler metric whose curvature and its covariant derivatives are bounded. Then M admits a Kähler-Einstein metric which is uniformly equivalent to the above metric.

If M admits a Hermitian metric with strongly negative Ricci curvature and is the increasing union of relatively compact, smooth, pseudoconvex open submanifolds, then there exists a unique (up to a scalar) almost complete Kähler-Einstein metric on M. Moreover, this metric is complete if M is complete.

In particular, there exists a complete Kähler-Einstein metric on any bounded domain in $\mathbf{C}^n$ which is the intersection of domains with C^2-boundaries. In the above statement, $\mathbf{C}^n$ can also be replaced by a Hermitian manifold with Ricci curvature bounded from above by a negative constant.

Mok and Yau [Mk-Y] proved that there exists a complete Kähler-Einstein metric on any bounded pseudoconvex domain in $\mathbf{C}^n$. This is the only known "canonical" metric on arbitrary bounded domains of holomorphy which is complete.

We now consider the case where the volume of M is finite. In this case, the "infinity" of M is very small (whereas the infinity of a bounded domain in $\mathbf{C}^n$ is quite large). The following is then conjectured: If the Ricci curvature is negative and M has finite topological type, then M can be compactified, that is, $M = \bar{M}/(\text{subvariety})$ for some compact Kähler manifold $\bar{M}$. In some cases, $\bar{M}$ is actually algebraic and hence M is quasi-projective.

For a locally Hermitian symmetric space M of finite volume, Baily and Borel [B-B], Satake [St] and Mumford [Mu] obtained (different) compactifications more or less explicitly. For these manifolds, Kähler-Einstein metrics exist. Siu and Yau [S-Y3] proved that a complete manifold, with finite volume with its curvature bounded between two negative constants, is quasi-projective.

If the above conjecture is true, then in studying Kähler manifolds with finite volume (and bounded covariant derivatives of the curvature) one need only consider $\bar{M}\backslash(D_1 \cup \cdots \cup D_k)$ where $\bar{M}$ is a compact Kähler manifold and $D_1, \dots, D_k$ are connected divisors. If we have suitable algebraic data on how D_i looks like and how D_i intersects D_j, then one hopes that one may be able to construct Kähler-Einstein metrics on M. In dimension two, this is well understood. For example, suppose $C \subseteq \bar{M}^2$ is an elliptic curve and $C \cdot C < 0$. If s is a section of the bundle $[C]$ and $C = \{s=0\}$ then $dv_{\bar{M}}/|s|^2(\log|s|^2)^3$ is a complete asymptotic Kähler-Einstein metric on $\bar{M}/C$ with C as the cusps of the metric.

Suppose that D is a divisor on a compact Kähler manifold M satisfying $c_1(K+[D]) \geqslant 0$ on $\bar{M}$, $c_1(K+[D]) > 0$ on $\bar{M}\backslash D$ and $(K+[D]) - \varepsilon[D]\,|_D > 0$ then $\bar{M}\backslash D$ admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold $M = \bar{M}\backslash D$, a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes $\tilde{c}_i(M, D)$. The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes $\tilde{c}_1$ and $\tilde{c}_2$:

$$(*') \qquad (-1)^n \tilde{c}_1^{n-2} \cdot \tilde{c}_2 \geqslant \frac{(-1)^n}{2(n+1)} \tilde{c}_1^n .$$

A particularly significant fact is that equality holds in (*) if the quasi-projective manifold $\bar{M}\backslash D$ is the quotient of the unit ball in $\mathbf{C}^n$.

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \geqslant dv_{\text{Kähler-Einstein}} \geqslant c_2 dv_{\text{Caratheodory}}$$

where c_1 and c_2 are two universal positive constants. We have the following question: If the Caratheodory metric of M is complete, does M admit a complete Kähler-Einstein metric?

6. Ricci Flat Metrics on Noncompact Manifolds

We now consider Ricci flat metrics on a complete, noncompact manifold M. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose g and g' are two Ricci flat Kähler metrics on M. If they satisfy $g_{i\bar{j}} - g'_{i\bar{j}} = \partial\bar{\partial}F$ with F bounded, then $g_{i\bar{j}} = g'_{i\bar{j}}$. Note that in the compact case, the above condition means that g and g' belong to the same Kähler class. It also may be possible to drop the condition that F is bounded since there do not exist too many Ricci flat metrics.