3. Hermitian Manifolds and Stable Vector Bundles

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3. Hermitian Manifolds and Stable Vector Bundles

We will consider canonical metrics on compact complex manifolds which are not necessarily Kählerian. For Hermitian manifolds in general, it is difficult to find canonical metrics because the Hermitian connection has torsion and hence is not Riemannian. Therefore one would like to assume extra conditions on M. Let g be a Hermitian metric on M and ω its Kähler form. One natural condition is to assume that

$$\partial\bar{\partial}(\omega^{n-1}) = 0\,, \tag{1}$$

which is weaker than the condition of being Kähler. One would like to put more conditions on g, besides (1), to make the metric more canonical. Motivated by the theory of supersymmetry, Hull and Witten [HW] proposed the following condition on ω. Locally one should be able to write ω as $\partial\theta + \bar{\partial}\,\bar{\theta}$ where θ is a (0, 1) form. Notice that if ω is Kähler, it can always be written as $\partial\bar{\partial}f$.

Let us now demonstrate that the above condition is equivalent to the condition $\partial\bar{\partial}\omega = 0$. Clearly, we have only to prove the condition $\partial\bar{\partial}\omega = 0$ implies that ω can be written in the above form. As $\bar{\partial}\omega$ is a closed form, it is locally exact. By comparting the types, we can find a (0, 2) form Ω and a (1, 1) form ω, so that $\bar{\partial}\omega = \partial\Omega + \bar{\partial}\omega'$ with $\bar{\partial}\Omega = 0$ and $\partial\omega' = 0$. Noticing that $\omega = \bar{\omega}$, we can then prove that $\omega - \omega' - \bar{\omega}' - \Omega - \bar{\Omega}$ is a closed form. Therefore, locally it is exact and we can find a (0, 1) form so that $\omega - \omega' - \bar{\omega}' = \partial\theta + \bar{\partial}\,\bar{\theta}$. Since $\partial\omega' = 0$, locally ω' is ∂-exact and we have proved locally ω is the form that we seek.

Recently Todorov observed that any compact complex manifold admits a Hermitian form ω with $\partial\bar{\partial}\omega = 0$. Therefore it seems that for any compact complex manifold, it is of interest to study the group obtained by taking the quotient of (1, 1) form ω with $\partial\bar{\partial}\omega = 0$ by the subgroup cosets of $\partial\theta + \bar{\partial}\,\bar{\theta}$ where θ is globally defined (0, 1) form.

Now let V be a holomorphic vector bundle over a compact manifold M with the property $\partial\bar{\partial}(\omega^{n-1}) = 0$. We can define the degree of the bundle V with respect to ω by

$$\deg_\omega V = \int_M \Xi_1(V) \wedge \omega^{n-1}\,;$$

where $\Xi_1(V)$ denotes the Ricci form of the bundle V. Since $\partial\bar{\partial}(\omega^{n-1}) = 0$, this definition is independent of the choice of metric on V.

In [U-Y], Uhlenbeck and Yau proved the following:

(2) Suppose V is a holomorphic vector bundle over a compact Kähler manifold M. If V is stable, i.e., $\dfrac{\deg_\omega V'}{\operatorname{rank} V'} < \dfrac{\deg_\omega V}{\operatorname{rank} V}$ for every coherent subsheaf $V' \subseteq V$ such that $0 < \operatorname{rank}(V') < \operatorname{rank}(V)$, then there exists a Hermitian-Einstein metric on V which is unique up to a constant.

Conversely, the existence of a Hermitian-Einstein metric on V implies that V is direct sum of stable bundles. This was proved by Kobayashi and Lübke [Lu]. Moreover, it is likely that the condition M be Kähler can be replaced by (1). It should be noted that the above theorem was proved by Donaldson [D2] for algebraic surfaces.

We now state some corollaries of (2). First of all, the symmetric tensor product bundle of a stable holomorphic vector bundle is also stable. Secondly, if V is a stable bundle, then for $r = \operatorname{rank}(V)$,

(3) $$\int_M (2r\, c_2(V) - (r-1)c_1^2(V)) \wedge \omega^{n-2} \geqslant 0,$$

and equality holds if and only if up to finite cover of M, V is a direct sum of line bundles (when $n = 2$, this was due to Bogomolov [Bo]) without dealing with the case of equality. Therefore, if $c_1^2(V) = 0$ then $\int_M c_2(V) \wedge \omega^{n-2} \geqslant 0$ and equality holds if and only if V is flat and unique up to a scalar. These results are in fact generalizations of those in the Riemann surface case. In particular, let V be a holomorphic vector bundle over a Riemann surface Σ_g. Then V is stable and $c_1(V) = 0$ if and only if there exists a Hermitian metric on V with zero curvature, i.e., if and if there is a unitary representation of $\pi_1(\Sigma_g)$ (see Narashimhan and Seshadri [N-S] for details.

We now consider the moduli space of stable vector bundles. Let $M(r, d)$ be a complete family of stable vector bundles of fixed rank r and fixed degree d over a Riemann surface Σ_g. Can one prove that $c_1(M_g) > 0$, in particular, can one construct a Kähler metric on M_g with positive Ricci curvature? Cho [Co] proved that there exists a Kähler metric on $M_g(r, d)$ with nonnegative holomorphic sectional curvature. However, even the positivity of the holomorphic sectional curvature does not imply the positivity of the Ricci curvature. For example, let H be the hyperplane bundle over $\mathbf{CP}^1$ and (1) the trivial line bundle. Then the Hirzebruch surfaces M_d

$= \mathbf{P}(H^d + (1))$ have Kähler metrics with positive holomorphic sectional curvature. On the other hand, for $d \geqslant 3$, M_d does not have positive first Chern class.

4. Chern Number Inequalities

In 1976, the author proved the Calabi conjecture and demonstrated the following Chern number inequality for algebraic manifolds with either ample or trivial canonical line bundles:

$$(*) \qquad (-1)^n c_2 c_1^{n-2} \geqslant \frac{(-1)^n}{2(n+1)} c_1^n$$

where equality holds if and only if M is covered by the ball, i.e., $M = B/\Gamma$ for some $\Gamma \subseteq SU(n, 1)$. Around the same time, Miyaoka [M3], extending the method of Bogomolov, obtained the same inequality for $n = 2$ under the weaker assumption that the Kodaira dimension of the surface is non-negative. However, he has not shown that equality holds if and only if M is covered by the ball.

By studying surfaces with singularities, Cheng and Yau [C-Y2] proved inequality (*) for surfaces of general type (equality holds if and only if M^2 is covered by the ball). The arguments in [C-Y2] can also be generalized to higher dimensions. One can also characterize surfaces M which are biholomorphic to B^n/Γ where $\Gamma \subseteq SU(2, 1)$ is allowed to have fixed points. Note that M is, in general, a variety since Γ may have fixed points.

It is also interesting to study manifolds which satisfy certain Chern number inequalities. Surfaces which satisfy inequality (*) have been studied by Hirzebruch, Deligne, Mostow, etc. A corollary of [Y2] is the following rigidity theorem for Kählerian structures on $\mathbf{CP}^n$: The only Kählerian structure on $\mathbf{CP}^n$ is the standard one; moreover, the only complex structure on $\mathbf{CP}^2$ is the standard one. For n odd, this result was due to Hirzebruch and Kodaira [H-K].

We now sketch the proof of inequality (*) when the canonical line bundle K of M is ample. In this case, there exists a Kähler-Einstein metric on K. For Kähler-Einstein metrics one observes that the Chern integral associated to the left hand side of (*) can be expressed in terms of the length squared of the curvature tensor. Since the Ricci tensor is the only part of the curvature tensor, the right hand side, which can be written as the determinant of the Ricci tensor, can be dominated by the left hand side.