§6. Canonical Metrics Over Complex Manifolds

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **01.06.2024**

analyticity of the harmonic map f. However, it seems to be difficult to decide which cycles can be represented by continuous images of Kähler manifolds.

§ 6. CANONICAL METRICS OVER COMPLEX MANIFOLDS

Given a complex manifold M, one could like to find "canonical" metrics on M so that one can produce invariants for the complex structure. One natural requirement for canonical metrics is that the totality of them can be parametrized by a finite dimension space and that they be invariant under the group of biholomorphisms.

1. THE BERGMAN, KOBAYASHI-ROYDEN AND CARATHEODORY METRICS

The Bergman metric was first introduced as a natural metric defined on bounded domains in $\mathbf{C}^n$. Later, the definition was generalized to complex manifolds whose canonical bundle K admit sufficiently many sections. For a domain D in $\mathbf{C}^n$, let $H^2(D)$ denote the space of square integrable holomorphic functions of D. Choose an orthonormal basis $\{\phi_i\}$ of this space. Then the Bergman kernel is defined as

$$K(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w)\,.$$

Notice that the definition of the Bergman kernel is independent of the choice of orthonormal basis. Moreover, K is holomorphic in the variables z and $\bar{w}$.

We can now define the Bergman metric by

$$ds^2 = \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)\, dz_i \otimes d\bar{z}_j\,.$$

The naturality of the Bergman metric can easily be seen from the definition of the Bergman kernel. Let D_1 and D_2 be two domains in $\mathbf{C}^n$, and $K_1(z, w)$ and $K_2(z', w')$ their respective Bergman kernels. If $F : D_1 \to D_2$ is a biholomorphism, then K_1 and K_2 are related by the formula

$$K_1(z, w) = K_2\big(f(z), f(w)\big) \det\left(\frac{\partial F}{\partial z}\right) \overline{\det\left(\frac{\partial F}{\partial w}\right)}\,.$$

If the canonical bundle K of M admits enough global, square integrable sections, we can choose an orthonormal basis $\{\phi_i\}$ of sections which will give rise to an embedding $F: M \to \mathbf{CP}^k$. The pull-back metric $F^*(ds^2)$ is the Bergman metric of M. This definition agrees with the previous definition of the Bergman metric when M is a complex domain because any holomorphic function over D can be though of as a section of K.

Intuitively speaking, a complete understanding of the Bergman metric would give us a clear picture of the geometry of the automorphisms of a domain; it would also provide us with a lot of invariants of the domain. In the past few years there has been a lot of progress based on Fefferman's work [Fe]. Fefferman looked at the asymptotic behavior of $K(z, z)$ near the boundary of a domain. Roughly, he proved that the Bergman kernel has the following expansion along the diagonal.

$$K(z, z) = \phi(z)/\Psi^{n+1}(z) + \tilde{\phi}(z) \log \Psi(z)$$

where $\phi, \tilde{\phi} \in C^\infty(D)$, $\phi|_{\partial D} = 0$, and Ψ is the defining function for the domain D.

Moreover, near the boundary we have

$$K(z, w) = \phi(z, w)/\Psi^{n+1}(z, w) + \tilde{\phi}(z, w) \log \Psi(z, w)$$

where $\phi(z, w)$, $\tilde{\phi}(z, w)$ and $\Psi(z, w)$ are extensions of ϕ, $\tilde{\phi}$ and Ψ, respectively, which satisfy certain conditions.

One would actually like to know more about the boundary behavior of the Bergman kernel and metric, the behavior of the curvature of the metric, and other related geometric properties of the metric when Ω is not smooth. Let Ω be a manifold and ds^2_Ω the Bergman metric. If Ω admits a properly discontinuous group of automorphisms we can consider the quotient manifold Ω/Γ and pull-back its Bergman metric $ds^2_{\Omega/\Gamma}$ to Ω. Kazhdan [Kz] proved that if the discrete automorphism group Γ of Ω has a filtration $\Gamma \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n \supseteq \cdots$ with $[\Gamma_i, \Gamma_{i+1}] < \infty$ and $\bigcap_i \Gamma_i = (1)$, then the pull-backs of the Bergman metrics $ds_i^2 = ds^2_{\Omega/\Gamma_i}$ will converge on Ω to the Bergman metric ds^2_Ω of Ω.

Another interesting direction is to look at the global sections of the powers of the canonical bundle. Consider $H^0(M, K^r)$ for r sufficiently large; a choice of basis gives a map $\phi_r: M \to \mathbf{P}(H^2(M, K^r))$. Taking the $1/r$ multiple of the restriction of Fubini-Study metric of $\mathbf{P}(H^2(M, K^r))$, one has a sequence of metrics on M. One would like to know if, as r tends to infinity, a limiting metric exists. If such a metric does exist, it should be "canonical" and hopefully Kähler-Einstein.

For a complex manifold Ω there are two other intrinsically defined pseudometrics: the Kobayashi-Royden metric and the Caratheodory metric. Let Δ be the Poincaré disk in $\mathbf{C}$. We denote by $\Delta(\Omega)$ the set of holomorphic maps from Ω to Δ, $\Omega(\Delta)$ the set of holomorphic maps from Δ to Ω. Fix the Poincaré distance on Δ. The Caratheodory metric is defined by

$$F_\Omega : T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_\Omega(z, z) = \sup\{| f_*(z) | : f \in \Delta(\Omega), f(z) = 0\}.$$

The Kobayashi-Royden metric on Ω is defined by

$$F_k : T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_k(z, \xi) = \inf\{| u | : f \in \Omega(\Delta), f(0) = z, f_*(u) = \xi\}.$$

Clearly, these two intrinsically defined metrics are distance decreasing under holomorphic maps and invariant under biholomorphic maps.

B. Wong [Wo1] has shown that the holomorphic sectional curvature of the Caratheodory metric is less than or equal to -4, whereas the holomorphic sectional curvature of the Kobayashi metric is not less than -4 when the metric is nontrivial (for the Bergman metric, it is known that the holomorphic sectional curvature is not greater than 4). However, one disadvantage of these two metrics is that they are neither bilinear nor smooth on the tangent spaces (F is only upper-semicontinuous in general).

In some special cases we have a better understanding of these two metrics. For example, a manifold with strongly negative holomorphic sectional curvature always admits a nontrivial Kobayashi-Royden metric. The major theorem in this subject is due to Royden who showed that the Kobayashi-Royden metric is actually the Teichmüller metric. It is a curious fact that the Teichmüller metric has constant holomorphic sectional curvature. Can we classify those complex manifolds that admit Finsler metric with constant holomorphic sectional curvature?

Lempert [Le1], [Le2] proved that the Kobayashi and Caratheodory metrics are actually the same for convex domains in $\mathbf{C}^n$. By using the existence of an extremal mapping, he constructed a lot of bounded holomorphic functions. His theory only works for convex domains; still, it is interesting to see how one can generalize his ideas or use these two metrics to construct bounded holomorphic functions on more general manifolds.

Another interesting fact, proved by B. Wong [Wo2], is that if a smooth, bounded domain in $\mathbf{C}^n$ covers a closed manifold, then it must be the unit ball. This partially confirms the conjecture that a bounded convex domain (not required to be smooth) which covers a closed manifold must be symmetric. His proof needed the boundary estimate of the Kobayashi and Caratheodory metrics.

In general, one would like to compare the Bergman, Kobayashi-Royden, Caratheodory metrics and the Kähler-Einstein metric discussed in the next section. We know that the Caratheodory metric is the smallest of the three. This can be seen by using the generalized Schwarz lemma for Kähler manifolds [Y4]. Yau (see the later improvement by Chan-Cheng-Lu) proved that if $f: M \to N$ is a holomorphic map where M is a complete Kähler manifold with Ricci curvature bounded from below by a constant and N is a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant, then f decreases distances up to a constant depending on the curvatures of M and N. Is this true if N is only a Finsler space? If it were true, then one expects that Teichmüller metric is uniformly equivalent to the Kähler-Einstein metric.

2. Kähler-Einstein Metrics on Compact Kähler Manifolds

Let M be a compact Kähler manifold. A necessary condition for the existence of a Kähler-Einstein metric on M is as follows.

(*) There exists a Kähler class Ω such that the first Chern class $c_1(M)$ is cohomologous to some real constant multiple of Ω.

This condition is equivalent to the following:

(*)′ The first Chern class satisfies $c_1(M) > 0$, $c_1(M) = 0$ or $c_1(M) < 0$.

It was proved by the author [Y1], [Y2] that when $c_1(M) = 0$ or $c_1(M) < 0$, (for the latter case see also Aubin [Au3]) there exists in every Kähler class a unique Kähler-Einstein metric. When $c_1(M) > 0$, the space Kähler-Einstein metrics are invariant under automorphism group. However, existence does not hold in general and one would like to impose conditions on M to ensure existence.

We now consider the obstruction, due to Futaki [Fu1], to the existence of Kähler-Einstein metrics when $c_1(M) > 0$; we also consider the notion of "extremal metrics" due to Calabi [Ca2]. Fix a Kähler class $\Omega = [\omega] \in H^{1,1}(M)$ on a compact Kähler manifold M and denote by H_Ω the space of all Kähler metrics with Kähler class Ω. Define the functional

$$F: H_\Omega \to \mathbf{R} \quad \text{by} \quad F: (g) \to \int_M R^2,$$

where R denotes the scalar curvature of the metric g. Calabi called a critical point of this functional an extremal metric. Any Kähler-Einstein metric minimizes $\int_M R^2$ in its Kähler class and hence is an extremal metric. This follows from the Schwarz inequality and the fact that $\int_M R$ is equal to $c_1(M) \cup \omega^{n-1}$ evaluated on the fundamental class of M, where ω is the Kähler form of g.

Calabi proved that for an extremal metric g, the gradient vector field $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic. He also proved that a decomposition theorem holds, analogous to that of Matsushima and Lichnerowicz for constant scalar curvature, for the automorphism group of M. In particular, he proved that X gives rise to a compact subgroup of Aut (M). Levine [Lv] gave an example of a compact surface M^2 with no compact connected subgroup in Aut (M); hence M^2 does not admit any Kähler-Einstein metrics.

For other examples of when Aut (M) is not reductive, see Sakane [Sk1], [Sk2], Ishikawa-Sukane [I-S] and Yau [Y3]. By the theorems of Calabi or Matsushima-Lichnerowicz, these examples do not admit any Kähler-Einstein metrics. Futaki [Fu1] also has constructed examples where Aut (M) is reductive and we will consider them later. So far, however, all examples of a Kähler manifold with positive first Chern class which does not admit a Kähler-Einstein metric admit nontrivial holomorphic vector field, it is natural to ask the following question: If there exists no nonzero holomorphic vector field on M, and if the tangent bundle of M is stable, can we always minimize the functional F? The motivation for the assumption on the stability will be discussed later. Of course, if the answer to the above question is yes, then (*) would also be a sufficient condition for the existence of Kähler-Einstein metrics.

In fact, suppose $c_1(M) = C[\omega]$ and g is an extremal metric. Since $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic, it follows that $X = 0$, R is constant and the Ricci form of g is a harmonic form representing $c_1(M)$. One concludes that $R_{i\bar{j}} = Cg_{i\bar{j}}$ from the uniqueness of harmonic forms in a cohomology class; hence g is a Kähler-Einstein metric. Calabi [Ca2] proved that, each extremal metric g is a local, nondegenerate point of the functional F. The metric g also exhibits the greatest possible degree of symmetry compatible with the complex structure of M. Let C_Ω denotes the set of extremal metrics in H_Ω, which is diffeomorphic to a finite dimensional Euclidean space.

Moreover, if one metric in C_Ω has constant scalar curvature, then every metric in C_Ω has constant scalar curvature. One expects that the only critical points of F are global minimums of F, form a connected set, and that the group of automorphisms of M which preserve the class Ω acts transitively on C_Ω.

We now consider Futaki's obstruction to the existence of a Kähler-Einstein metric on compact Kähler manifold M with $c_1(M) > 0$. Let $\eta(M)$ denote the Lie algebra of holomorphic vector fields of M, ω a Kähler form representing $c_1(M)$, and γ_ω its Ricci form which also represents $c_1(M)$. Then $\gamma_\omega = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial \log\det(g_{ij})$ and hence $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial G$ for some smooth function G. Define the character $f : \eta(M) \to \mathbf{C}$ by $f : X \to \int_M (XG) \cdot \omega^n$. Futaki proved that f is independent of the choice of representative ω of $c_1(M)$. Hence the integer $\delta_M = \dim(\eta(M)/\ker(f))$ depends only on the complex structure of M.

If M has a Kähler-Einstein metric then $\delta_M = 0$; Futaki conjectures that the converse is also true. This would be the case if Calabi's functional F attains a minimum. Since $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi} \partial\bar\partial G$, one has that $R = n + \Delta G$. Then $f(X) = \int (XG)\omega^n = \int (R^\alpha G_\alpha)\omega^n = \int |\Delta G|^2 \omega^n$; hence $\delta_M = 0$ implies that $G =$ constant, i.e., g is a Kähler-Einstein metric.

Using the obstruction δ_M, Futaki gave examples of compact Kähler manifolds with $c_1(M) > 0$, Aut (M) reductive, and $\delta_M = 1$. Hence, there does not exist Kähler-Einstein metrics on these examples. Let H_n denote the hyperplane bundle of $\mathbf{CP}^n$ and $\pi_n : H_n \to \mathbf{CP}^n$ the projection map ($n = 1, 2$). If we let $M^5 = \mathbf{P}(E)$ where $E = \pi_1^*(H_1) + \pi_2^*(H_2)$ is considered as a bundle over $\mathbf{CP}^2$, then M is such an example. The following is the lowest dimensional example. If $H \subseteq \mathbf{CP}^3$ is a hyperplane and $C \subseteq H$ a quadratic curve, then let M be $\mathbf{CP}^3$ blown up along C and at a point outside of H.

Futaki's idea is to construct an obstruction for the Ricci form to be harmonic. For the curvature forms representing the higher Chern classes, see Bando [B2]. For questions related to the character f, see Futaki [Fu2] and Futaki-Morita [F-M]. Bando also proved the uniqueness of Kähler-Einstein metric on M with $c_1(M) > 0$, up to holomorphic automorphisms of M.

3. Hermitian Manifolds and Stable Vector Bundles

We will consider canonical metrics on compact complex manifolds which are not necessarily Kählerian. For Hermitian manifolds in general, it is difficult to find canonical metrics because the Hermitian connection has torsion and hence is not Riemannian. Therefore one would like to assume extra conditions on M. Let g be a Hermitian metric on M and ω its Kähler form. One natural condition is to assume that

$$\partial\bar{\partial}(\omega^{n-1}) = 0\,, \tag{1}$$

which is weaker than the condition of being Kähler. One would like to put more conditions on g, besides (1), to make the metric more canonical. Motivated by the theory of supersymmetry, Hull and Witten [HW] proposed the following condition on ω. Locally one should be able to write ω as $\partial\theta + \bar{\partial}\bar{\theta}$ where θ is a (0, 1) form. Notice that if ω is Kähler, it can always be written as $\partial\bar{\partial}f$.

Let us now demonstrate that the above condition is equivalent to the condition $\partial\bar{\partial}\omega = 0$. Clearly, we have only to prove the condition $\partial\bar{\partial}\omega = 0$ implies that ω can be written in the above form. As $\bar{\partial}\omega$ is a closed form, it is locally exact. By comparting the types, we can find a (0, 2) form Ω and a (1, 1) form ω, so that $\bar{\partial}\omega = \partial\Omega + \bar{\partial}\omega'$ with $\bar{\partial}\Omega = 0$ and $\partial\omega' = 0$. Noticing that $\omega = \bar{\omega}$, we can then prove that $\omega - \omega' - \bar{\omega}' - \Omega - \bar{\Omega}$ is a closed form. Therefore, locally it is exact and we can find a (0, 1) form so that $\omega - \omega' - \bar{\omega}' = \partial\theta + \bar{\partial}\bar{\theta}$. Since $\partial\omega' = 0$, locally ω' is ∂-exact and we have proved locally ω is the form that we seek.

Recently Todorov observed that any compact complex manifold admits a Hermitian form ω with $\partial\bar{\partial}\omega = 0$. Therefore it seems that for any compact complex manifold, it is of interest to study the group obtained by taking the quotient of (1, 1) form ω with $\partial\bar{\partial}\omega = 0$ by the subgroup cosets of $\partial\theta + \bar{\partial}\bar{\theta}$ where θ is globally defined (0, 1) form.

Now let V be a holomorphic vector bundle over a compact manifold M with the property $\partial\bar{\partial}(\omega^{n-1}) = 0$. We can define the degree of the bundle V with respect to ω by

$$\deg_\omega V = \int_M \Xi_1(V) \wedge \omega^{n-1}\,;$$

where $\Xi_1(V)$ denotes the Ricci form of the bundle V. Since $\partial\bar{\partial}(\omega^{n-1}) = 0$, this definition is independent of the choice of metric on V.

In [U-Y], Uhlenbeck and Yau proved the following:

(2) Suppose V is a holomorphic vector bundle over a compact Kähler manifold M. If V is stable, i.e., $\frac{\deg_\omega V'}{\operatorname{rank} V'} < \frac{\deg_\omega V}{\operatorname{rank} V}$ for every coherent subsheaf $V' \subseteq V$ such that $0 < \operatorname{rank}(V') < \operatorname{rank}(V)$, then there exists a Hermitian-Einstein metric on V which is unique up to a constant.

Conversely, the existence of a Hermitian-Einstein metric on V implies that V is direct sum of stable bundles. This was proved by Kobayashi and Lübke [Lu]. Moreover, it is likely that the condition M be Kähler can be replaced by (1). It should be noted that the above theorem was proved by Donaldson [D2] for algebraic surfaces.

We now state some corollaries of (2). First of all, the symmetric tensor product bundle of a stable holomorphic vector bundle is also stable. Secondly, if V is a stable bundle, then for $r = \operatorname{rank}(V)$,

(3) $$\int_M (2r\, c_2(V) - (r-1)c_1^2(V)) \wedge \omega^{n-2} \geqslant 0,$$

and equality holds if and only if up to finite cover of M, V is a direct sum of line bundles (when $n = 2$, this was due to Bogomolov [Bo]) without dealing with the case of equality. Therefore, if $c_1^2(V) = 0$ then

$\int_M c_2(V) \wedge \omega^{n-2} \geqslant 0$ and equality holds if and only if V is flat and unique up to a scalar. These results are in fact generalizations of those in the Riemann surface case. In particular, let V be a holomorphic vector bundle over a Riemann surface Σ_g. Then V is stable and $c_1(V) = 0$ if and only if there exists a Hermitian metric on V with zero curvature, i.e., if and if there is a unitary representation of $\pi_1(\Sigma_g)$ (see Narashimhan and Seshadri [N-S] for details.

We now consider the moduli space of stable vector bundles. Let $M(r, d)$ be a complete family of stable vector bundles of fixed rank r and fixed degree d over a Riemann surface Σ_g. Can one prove that $c_1(M_g) > 0$, in particular, can one construct a Kähler metric on M_g with positive Ricci curvature? Cho [Co] proved that there exists a Kähler metric on $M_g(r, d)$ with nonnegative holomorphic sectional curvature. However, even the positivity of the holomorphic sectional curvature does not imply the positivity of the Ricci curvature. For example, let H be the hyperplane bundle over $\mathbf{CP}^1$ and (1) the trivial line bundle. Then the Hirzebruch surfaces M_d

$= \mathbf{P}(H^d + (1))$ have Kähler metrics with positive holomorphic sectional curvature. On the other hand, for $d \geqslant 3$, M_d does not have positive first Chern class.

4. Chern Number Inequalities

In 1976, the author proved the Calabi conjecture and demonstrated the following Chern number inequality for algebraic manifolds with either ample or trivial canonical line bundles:

$$(*) \qquad (-1)^n c_2 c_1^{n-2} \geqslant \frac{(-1)^n}{2(n+1)} c_1^n$$

where equality holds if and only if M is covered by the ball, i.e., $M = B/\Gamma$ for some $\Gamma \subseteq SU(n, 1)$. Around the same time, Miyaoka [M3], extending the method of Bogomolov, obtained the same inequality for $n = 2$ under the weaker assumption that the Kodaira dimension of the surface is non-negative. However, he has not shown that equality holds if and only if M is covered by the ball.

By studying surfaces with singularities, Cheng and Yau [C-Y2] proved inequality (*) for surfaces of general type (equality holds if and only if M^2 is covered by the ball). The arguments in [C-Y2] can also be generalized to higher dimensions. One can also characterize surfaces M which are biholomorphic to B^n/Γ where $\Gamma \subseteq SU(2, 1)$ is allowed to have fixed points. Note that M is, in general, a variety since Γ may have fixed points.

It is also interesting to study manifolds which satisfy certain Chern number inequalities. Surfaces which satisfy inequality (*) have been studied by Hirzebruch, Deligne, Mostow, etc. A corollary of [Y2] is the following rigidity theorem for Kählerian structures on $\mathbf{CP}^n$: The only Kählerian structure on $\mathbf{CP}^n$ is the standard one; moreover, the only complex structure on $\mathbf{CP}^2$ is the standard one. For n odd, this result was due to Hirzebruch and Kodaira [H-K].

We now sketch the proof of inequality (*) when the canonical line bundle K of M is ample. In this case, there exists a Kähler-Einstein metric on K. For Kähler-Einstein metrics one observes that the Chern integral associated to the left hand side of (*) can be expressed in terms of the length squared of the curvature tensor. Since the Ricci tensor is the only part of the curvature tensor, the right hand side, which can be written as the determinant of the Ricci tensor, can be dominated by the left hand side.

If equality holds for (*), one sees that the integrands of both sides are equal. This last fact turns out to be equivalent to M having constraint holomorphic sectional curvature. Hence equality holds in (*) if and only if M is covered by the ball.

Kähler-Einstein metrics do not exist on algebraic manifolds whose canonical line bundle is not a multiple of some ample line bundle. However, it is still possible to study the inequality (*) for algebraic manifolds whose canonical line bundle is almost ample. In [Y1] it was proven that there exists a Kähler-Einstein metric which is degenerate along the divisor where the canonical line bundle is trivial. Similarly one can require the metric to blow up in a certain way. This fact was used by Cheng and Yau [C-Y2] to prove the inequality (*) for surfaces of general type.

(**) $c_1(M) \leqslant 0$ on M, and $c_1(M) < 0$ outside a subvariety of M.

Recall that the Kodaira dimension $K(M)$ is defined by

$$K(M) = \begin{cases} -\infty & \text{if} \quad N(M) = 0 \\ \max \dim \{\phi_{mk}\}(M) & \text{if} \quad N(M) \neq 0 \end{cases},$$

where $N(M) = \{m > 0 \mid H^0(M, K^m) = 0\}$ and ϕ_{mk} is the pluricanonical mapping. It is easy to see that $K(M) \leqslant$ the algebraic dimension of $M \leqslant n$. If $K(M) = n$, then M is called a manifold of general type.

In dimension two, surfaces can be classified bimeromorphically by their Kodaira dimension. The surfaces with $K(M) = -\infty$, 0 or 1 are well understood; moreover, $K(M) = 2$ (i.e., M is a surface of general type) if and only if M satisfies (**). Suppose M is a three-fold of general type and K is the canonical line bundle divisor. Kawatama [Ka] proved that if $K \cdot C \leqslant 0$ for every algebraic curve $C \subseteq M$, then M satisfies (**).

Most likely (**) always implies (*); that is, if M^n is an algebraic manifold with almost ample canonical line bundle, then the inequality (*) holds. This is not known for $n \geqslant 3$. One would also like to know what the relationship is between manifolds of general type and the inequality (**). In this respect, consider the following theorem of Siu [S5]. First recall that Siegel's theorem [Sg] says that for a complex manifold M^n, the transcendence degree of the meromorphic function field of M over $\mathbf{C}$ is less than or equal to n. When equality holds, M is called a Moishezon manifold. A Moishezon manifold can always be obtained by blowing up and down an algebraic manifold a finite number of times and hence is birational to some projective algebraic manifold. For a Moishezon manifold, there always exists a holomorphic vector bundle L over M such that

$c_1(L) \geqslant 0$ on M and $c_1(L) > 0$ outside some subvariety of M. Siu [S5] proved that the converse is also true under the weaker assumption that $c_1(L)$ is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (**) is Moishezon. It is also not known whether $\mathbf{CP}^n$, $n \geqslant 4$, can admit a nonstandard structure which is Moishezon. For $n = 3$, T. Peternell [Pe] proved that if M is a Moishezon 3-fold which is topologically isomorphic to $\mathbf{CP}^3$, then M is the standard $\mathbf{CP}^3$. His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological $\mathbf{CP}^n$.

5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let g be a complete Kähler-Einstein metric on M^n, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$ for some constant c. If $c > 0$, Myer's theorem would imply N is compact. Hence, $c \leqslant 0$ and $c_1(M) \leqslant 0$. In this section we consider the case $c_1(M) < 0$ and leave the case $c_1(M) = 0$ for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics $g_{i\bar{j}}$ with $R_{i\bar{j}} = -g_{i\bar{j}}$. In particular, one would like to impose conditions on M to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if M and N are complete Kähler-Einstein manifolds with $R = -1$ and $F: M \to N$ is a biholomorphism, then F is an isometry. To prove this, let g and dv and g' and dv' denote the Kähler-Einstein metrics and volume forms of M and N, respectively. If we let $\rho = \log(F^*dv'/dv)$, then $\partial\bar{\partial}\rho = -f^*\operatorname{Ric}' + \operatorname{Ric} = F^*g' + g$. Taking traces, we have $\Delta\rho = -n + n \cdot e^{\rho/n}$. Hence, the maximum principle implies $\rho \leqslant 0$ and $F^*dv' \leqslant dv$. Replacing F by F^{-1}, we have $F^*dv' \geqslant dv$ and F is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric ds^2 on M is said to be almost complete if we can write M as an increasing union of domains Ω_α and there exist complete metrics ds_α^2 on Ω_α for each α such that ds_α^2 converges to ds^2 on compact subsets of M. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of M. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than a negative constant on M, then the pseudomeasure in the sense of Eiseman is in fact a measure, that is to say, M is measure hyperbolic.

In [C-Y1], Cheng and Yau obtained the existence of Kähler-Einstein metrics on a large class of noncompact manifolds. More precisely, they proved the following. Let M^n be a Hermitian manifold whose Ricci tensor defines a Kähler metric whose curvature and its covariant derivatives are bounded. Then M admits a Kähler-Einstein metric which is uniformly equivalent to the above metric.

If M admits a Hermitian metric with strongly negative Ricci curvature and is the increasing union of relatively compact, smooth, pseudoconvex open submanifolds, then there exists a unique (up to a scalar) almost complete Kähler-Einstein metric on M. Moreover, this metric is complete if M is complete.

In particular, there exists a complete Kähler-Einstein metric on any bounded domain in $\mathbf{C}^n$ which is the intersection of domains with C^2-boundaries. In the above statement, $\mathbf{C}^n$ can also be replaced by a Hermitian manifold with Ricci curvature bounded from above by a negative constant.

Mok and Yau [Mk-Y] proved that there exists a complete Kähler-Einstein metric on any bounded pseudoconvex domain in $\mathbf{C}^n$. This is the only known "canonical" metric on arbitrary bounded domains of holomorphy which is complete.

We now consider the case where the volume of M is finite. In this case, the "infinity" of M is very small (whereas the infinity of a bounded domain in $\mathbf{C}^n$ is quite large). The following is then conjectured: If the Ricci curvature is negative and M has finite topological type, then M can be compactified, that is, $M = \bar{M}/(\text{subvariety})$ for some compact Kähler manifold $\bar{M}$. In some cases, $\bar{M}$ is actually algebraic and hence M is quasi-projective.

For a locally Hermitian symmetric space M of finite volume, Baily and Borel [B-B], Satake [St] and Mumford [Mu] obtained (different) compactifications more or less explicitly. For these manifolds, Kähler-Einstein metrics exist. Siu and Yau [S-Y3] proved that a complete manifold, with finite volume with its curvature bounded between two negative constants, is quasi-projective.

If the above conjecture is true, then in studying Kähler manifolds with finite volume (and bounded covariant derivatives of the curvature) one need only consider $\bar{M}\backslash(D_1 \cup \cdots \cup D_k)$ where $\bar{M}$ is a compact Kähler manifold and $D_1, \ldots, D_k$ are connected divisors. If we have suitable algebraic data on how D_i looks like and how D_i intersects D_j, then one hopes that one may be able to construct Kähler-Einstein metrics on M. In dimension two, this is well understood. For example, suppose $C \subseteq \bar{M}^2$ is an elliptic curve and $C \cdot C < 0$. If s is a section of the bundle $[C]$ and $C = \{s=0\}$ then $dv_{\bar{M}}/|s|^2(\log|s|^2)^3$ is a complete asymptotic Kähler-Einstein metric on $\bar{M}/C$ with C as the cusps of the metric.

Suppose that D is a divisor on a compact Kähler manifold M satisfying $c_1(K+[D]) \geqslant 0$ on $\bar{M}$, $c_1(K+[D]) > 0$ on $\bar{M}\backslash D$ and $(K+[D]) - \varepsilon[D]|_D > 0$ then $\bar{M}\backslash D$ admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold $M = \bar{M}\backslash D$, a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes $\tilde{c}_i(M, D)$. The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes $\tilde{c}_1$ and $\tilde{c}_2$:

$$(*') \qquad (-1)^n \tilde{c}_1^{n-2} \cdot \tilde{c}_2 \geqslant \frac{(-1)^n}{2(n+1)} \tilde{c}_1^n .$$

A particularly significant fact is that equality holds in (*) if the quasi-projective manifold $\bar{M}\backslash D$ is the quotient of the unit ball in $\mathbf{C}^n$.

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \geqslant dv_{\text{Kähler-Einstein}} \geqslant c_2 dv_{\text{Caratheodory}}$$

where c_1 and c_2 are two universal positive constants. We have the following question: If the Caratheodory metric of M is complete, does M admit a complete Kähler-Einstein metric?

6. Ricci Flat Metrics on Noncompact Manifolds

We now consider Ricci flat metrics on a complete, noncompact manifold M. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose g and g' are two Ricci flat Kähler metrics on M. If they satisfy $g_{i\bar{j}} - g'_{i\bar{j}} = \partial\bar{\partial}F$ with F bounded, then $g_{i\bar{j}} = g'_{i\bar{j}}$. Note that in the compact case, the above condition means that g and g' belong to the same Kähler class. It also may be possible to drop the condition that F is bounded since there do not exist too many Ricci flat metrics.

In any case, the uniqueness problem is far from solved. Even when $M = \mathbf{C}^n$, Calabi proposed the following open problem: If $u: \mathbf{C}^n \to R$ is a strictly plurisubharmonic function with $\det\left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = 1$, then if the Kähler metric $ds_u^2 = \sum \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j$ is complete does it have zero curvature? Note that ds_u^2 is not complete in general. For example, Fatou and Bieberbach (see the book of Bochner and Martin [B-M], p. 45) gave a biholomorphism $F: \mathbf{C}^2 \to \Omega$, where $\Omega \subseteq \mathbf{C}^2$ is open and $\mathbf{C}^2/\Omega$ contains an open set, such that the Jacobian of F is identically equal to one. For $u = |z^1|^2 + |z^2|^2$, $ds^2_{u.F} = F^* ds_u^2 = F^* ds_0^2$ is not complete.

There are a lot of biholomorphisms F in $\mathrm{Aut}(\mathbf{C}^2)$ with Jacobian equal to one; for example, let $F(z, w) = (z + f(w), w)$ for any entire function f. For the above u, $u \circ F$ is still strictly plurisubharmonic and $ds_{u.F}$ is complete and Ricci flat. Thus, intuitively, the larger the group $\mathrm{Aut}(M)$, the more difficult the problem is.

We now consider the question of existence. Just as in the case of negative scalar curvature, the existence of a complete, Ricci flat, Kähler metric will impose restrictions on the complex structure of M. For example, by the Schwarz lemma [Y4], we know that there does not exist any nontrivial holomorphic maps from M to a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant. As a corollary, if there exists a nontrivial holomorphic map from $\bar{M}$ to an algebraic curve of genus greater than one, then $M \subseteq \bar{M}$ cannot admit any complete Kähler metric with nonnegative Ricci curvature.

We conjecture that if M^n admits a complete Ricci flat Kähler metric, then $M = \bar{M}\backslash(\text{divisor})$ where $\bar{M}$ is compact and Kähler. This would mean that the infinity of M cannot be too large. Now suppose $M^2 = \bar{M}\backslash(\text{divisor})$ and dv is a Ricci flat volume form on M. One would like to determine M; by going to the universal cover, we can assume M is simply connected. Locally, $dv = (\sqrt{-1})^2 k dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2$ for some positive real function k. Since $\mathrm{Ric}(dv) = 0$, we have $\partial\bar{\partial}(\log k) = 0$ and k can be written as $k = |h|^2$ for some locally defined holomorphic function h. By a monodromy argument, we obtain a holomorphic 2-form $\eta = h dz^1 \wedge dz^2$, with h nowhere zero and $\eta \wedge \bar{\eta} = dv$. Hence $\eta^{-1} = h^{-1} dz^1 \wedge dz^2$ can be considered as a global section of the anti-canonical bundle K^{-1}.

Intuitively, one might expect that h approaches ∞ near the infinity of M and η^{-1} can be extended to $\bar{M}$, that is, there exists a nontrivial section

$S \in H^0(\bar{M}, K^{-1})$. This would imply that either K is trivial on M or $H^0(\bar{M}, K^n) = 0$ for every $m > 0$ and hence the Kodaira dimension of $\bar{M}^2$ would either be $-\infty$ or 0. This is because if $t \in H^0(\bar{M}, K^n)$, then $t \cdot S^n$ is a holomorphic function on M and hence constant; since S is zero somewhere unless K is trivial, we have $t \cdot S^n = 0$, so that $t = 0$ unless K is trivial on M.

Since M is Kähler and simply connected, the minimal model of $\bar{M}$ is a Kähler surface with $K = 0$ or $-\infty$ and $b_1 = 0$. When $K = 0$, it is either a $K-3$ surface or Enriques' surface. When $K = -\infty$ it is either a rational surface or a ruled surface of genus zero, $\bar{M}^2$ is equal the minimal model blown up successively at a finite number of points, and $M = \bar{M} \backslash \{s=0\}$ for some $0 \neq s \in H^0(\bar{M}, K^{-1})$. Conversely, if $M = \bar{M} \backslash \{s=0\}$ with $s \in H^0(\bar{M}, K^{-1})$ and $\bar{M}$ is as above, then M should admit a Ricci flat, complete, Kähler metric. In higher dimensions, the situation is much more complicated.

In physics, the following question has been studied. Is a Ricci flat metric with a suitable locally asymptotic property actually unique? This is the case when the metric is asymptotically flat. One would also like to know what happens when the metric is locally asymptotic to a cone. Perhaps assuming that the metric is Kähler may make this problem easier.

The existence of Ricci flat metrics has many applications. For example, using Ricci flat metrics, Siu [S1] proved that any surface M^2 with $c_1(M) = 0$ and $H^1(M, \mathbf{R}) = 0$ must be Kähler. See also Todorov [To] for higher dimensions. One can also ask the following question: Let M^{2n} be a simply-connected, compact, complex manifold where $n \geqslant 2$. If there exists a non-degenerate 2-form $\omega \in H^{2,0}(M)$, is M then Kähler? Todorov claimed that M is Kähler under an additional assumption: $\dim H^{2,0}(M) = 1$.

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