4. Analytic objects

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **01.06.2024**

give some information about the topology besides the known inequality on Chern numbers.

4. Analytic objects

In order to understand the complex structure, it is important to understand the analytic objects attached to the structure. Here we give two examples:

A. *Holomorphic maps and vector bundles*

For a complex manifold M, the natural holomorphic vector bundles associated to it are TM, TM^*, $\Lambda^k TM$, $\otimes^k TM$, etc. Of special importance is the canonical line bundle $K = \Lambda^n TM^*$.

By blowing up points or submanifolds, one can get additional analytic objects. The Riemann-Roch theorem, which relates a topological invariant to an analytic invariant, is an important tool in constructing analytic objects or invariants from the given topological or analytic information.

The Yang-Mills theory is often useful in constructing holomorphic vector bundles and other objects over Kähler manifolds. Taubes [T1] used the anti-self-dual solutions to the Yang-Mills equations to construct holomorphic vector bundles of rank two over Kähler surfaces M^2. Is it possible to use this theory to recover the author's theorem that if M^2 is simply connected and its cup product is positive definite, then M^2 is biholomorphic to $\mathbf{CP}^2$?

Taubes [T2] also constructed holomorphic vector bundles over Kähler surfaces under the assumption of an inequality between the two Chern numbers (see also Donaldson [D1] and [D2]). So far, the above arguments only work in the two dimensional case. For higher dimensions, there is no good way to construct holomorphic vector bundles. The idea of Taubes can be extended to construct holomorphic vector bundles over high dimensional manifold. But it is not clear how large a class can one achieve in such a way.

B. *Analytic cycles*

Recall that by an analytic cycle, one simply means the formal sum of analytic subvarieties. Let M^n be an algebraic manifold and $V \subseteq M$ an analytic subvariety of codimension p. Then the fundamental cohomology class η_V of V belongs to $H^{p,p}(M) \cap H^{2p}(M; \mathbf{Z})$. Recall that an element

$\alpha \in H^{2p}(M; \mathbf{Q})$ is analytic if it can be represented by a linear combination, with rational coefficients, of the fundamental classes of subvarieties of codimension p, i.e., $\alpha = \sum_{i=1}^{k} b_i \eta_{V_i}$, where $b_i \in \mathbf{Q}$ and V_i is a subvariety of M. Clearly, every analytic element in $H^{2p}(M; \mathbf{Q})$ belongs to $H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$. Conversely, we have the Hodge Conjecture: Every element $\alpha \in H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$ is analytic. This is true when $p = 1$ and is called the Lefschetz theorem on (1, 1)-classes; it is not known for $p \geqslant 2$.

In a Kähler manifold M^n every analytic subvariety is area-minimizing. This follows in a straightforward way from the formulae of Wirtinger and Stokes. Conversely, under suitable conditions, area-minimizing submanifolds become subvarieties. For example, Siu-Yau [SY1] proved that if $f: \mathbf{CP}^1 \to M^n$ is energy-minimizing and the bisectional curvature of M is positive, then f is either holomorphic or anti-holomorphic.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M)$ group of projective transformations of $\mathbf{CP}^N$. Set

$$B(X, X) = \frac{d^2}{dt^2}(\mathrm{Vol}\, g_t(M))\bigg|_{t=0}, \quad \text{where} \quad X = \frac{dg_t}{dt}.$$

They proved that the trace of B is negative unless M is a subvariety.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M) \cap H^{2p}(M; \mathbf{Q})$, define a volume function as follows: $\mathrm{Vol}: PGL(N+1, \mathbf{C}) \to \mathbf{R}$ where $\mathrm{Vol}: g \to \inf_C \{\mathrm{Vol}_g(C) \mid C \text{ represents } \alpha\}$. Here α is the Poincaré dual of β and $\mathrm{Vol}_g(C)$ is the volume with respect to the metric $(g \circ f)^* ds_0^2$ where ds_0^2 denotes the Fubini-Study metric on $\mathbf{CP}^N$. If there exists a holomorphic C representing α, then $\mathrm{Vol}_g(\alpha) = \mathrm{Vol}_g(C)$ is independent of the choice of g. Hence Vol is a constant function which attains its minimum. On the other hand, if Vol has a minimum, then Lawson-Simons' argument shows that there exists a holomorphic C representing α. Therefore the Hodge conjecture would be proved if one could show the minimum of Vol is attained.

Siu [S2] obtained the following result. Let M be a compact Kähler manifold with strongly negative curvature. Then any element in $H_{2k}(M; \mathbf{Z})$, for $k \geqslant 2$, can be represented by an analytic subvariety if it can be represented by the continuous image of a compact Kähler manifold. His argument used the Bochner type formula for $\bar{\partial} f \wedge \partial \bar{f}$ to get the complex analyticity of the harmonic map f. However, it seems to be difficult to decide which cycles can be represented by continuous images of Kähler manifolds.

§ 6. Canonical Metrics Over Complex Manifolds

Given a complex manifold M, one could like to find "canonical" metrics on M so that one can produce invariants for the complex structure. One natural requirement for canonical metrics is that the totality of them can be parametrized by a finite dimension space and that they be invariant under the group of biholomorphisms.

1. The Bergman, Kobayashi-Royden and Caratheodory metrics

The Bergman metric was first introduced as a natural metric defined on bounded domains in $\mathbf{C}^n$. Later, the definition was generalized to complex manifolds whose canonical bundle K admit sufficiently many sections. For a domain D in $\mathbf{C}^n$, let $H^2(D)$ denote the space of square integrable holomorphic functions of D. Choose an orthonormal basis $\{\phi_i\}$ of this space. Then the Bergman kernel is defined as

$$K(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w) .$$

Notice that the definition of the Bergman kernel is independent of the choice of orthonormal basis. Moreover, K is holomorphic in the variables z and $\bar{w}$.

We can now define the Bergman metric by

$$ds^2 = \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)\, dz_i \otimes d\bar{z}_j .$$

The naturality of the Bergman metric can easily be seen from the definition of the Bergman kernel. Let D_1 and D_2 be two domains in $\mathbf{C}^n$, and $K_1(z, w)$ and $K_2(z', w')$ their respective Bergman kernels. If $F : D_1 \to D_2$ is a biholomorphism, then K_1 and K_2 are related by the formula

$$K_1(z, w) = K_2(f(z), f(w)) \det\left(\frac{\partial F}{\partial z}\right) \overline{\det\left(\frac{\partial F}{\partial w}\right)} .$$