§5. Kähler Geometry

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then the limiting S^2 will enclose a fake disk. Take a Jordan curve on this S^2 so that it decomposes the S^2 into two regions with equal area. Then one expects this Jordan curve to bound an embedded minimal disk in the fake disk. If one can achieve this, one can shrink the S^2 more and obtain a contradiction which will give a proof of the Poincaré conjecture.

In conclusion, minimal surface theory is surprisingly successful in being applied to three dimensional topology. I believe that a more thorough study of minimal surfaces will reveal more secrets about three manifolds.

§ 5. Kähler Geometry

In the following we consider four basic topics in complex geometry.

1. Existence of complex and almost complex structure.
2. Existence of Kähler and algebraic structures on complex manifolds.
3. Uniformization problems and the parametrization of metrics.
4. Analytic objects over complex manifolds, e.g., analytic cycles, holomorphic vector bundles, etc.

We will divide this section into four parts corresponding to these topics.

1. Complex and almost complex structures

Let M be an even dimensional oriented differentiable manifold. The existence of an almost complex structure J is equivalent to a reduction of the structure group of the tangent bundle from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. This is basically an algebraic problem and is well understood.

However, the question of when an almost complex structure is homotopic to an integrable almost complex structure (i.e., one which comes from a complex structure) is much harder. When $n = 1$, every M^2 admits an almost complex structure and every such structure is integrable and algebraic. For $n = 2$, ven de Van [V1] gave several examples of compact M^4's which admit an almost complex structure but not a complex structure. His argument is based on the computations of the first and second Chern classes. When $n \geqslant 3$, there are no such examples known so far. In particular, we do not know whether or not the almost complex manifold S^6 admits a complex structure. This problem has been open for a long time.

The topology of complex surfaces is not well understood. By the works of Donaldson, one may believe that every simply connected four dimensional compact manifold is the connected sum of algebraic surfaces. For nonsimply connected algebraic surfaces, it is more difficult to speculate. The basic problem is to find a way to construct complex structures. Perhaps one can ask the following question. Suppose M is a compact almost complex manifold satisfying $\chi(M) = 3\tau(M)$ and covered topologically by $\mathbf{R}^4$. (Here $\chi(M)$ is the Euler number and $\tau(M)$ is the index of M.) If every abelian subgroup of $\pi_1(M)$ is infinite cyclic, does M admit a complex structure so that M is covered holomorphically by the unit ball in $\mathbf{C}^2$? The Lefschetz theorem may be useful in the above question.

2. Kähler and algebraic structures

Let M^n be an n complex dimensional compact manifold with complex structure J. The first question is: When is J Kählerian, i.e., (M, J) admits a Kähler metric? Harvey-Lawson [H-L] gave an intrinsic characterization of the Kählerian condition if and only if M carries no positive currents which are the (1, 1)-components of boundaries. Hodge theory gives a lot of necessary conditions for complex manifolds to be Kähler. In particular, their even Betti numbers must be positive and their odd Betti numbers are even. Also, when (M, J) is Kählerian, its rational homotopy type is determined by its rational cohomology, see Deligne-Griffiths-Morgan-Sullivan [DGMS].

Now suppose M is a Kähler manifold, i.e., M has some Kählerian complex structure. When does M admit a non-Kählerian complex structure? When does M have a unique complex (or Kählerian) structure?

When $n = 2$, every compact complex surface with even first Betti number is Kählerian. (This follows from the classification of Kodaira because Miyaoka [M1] and Siu [S1] proved respectively that elliptic surfaces with even first Betti number and $K-3$ surfaces are Kählerian. From this one concludes that among the seven classes of surfaces in Kodaira's classification, the first five are Kählerian for every complex structure. The remaining two classes of surfaces have odd first Betti number and hence admit no Kähler metrics. In particular, one sees that on a Kähler surface M^2, all complex structures on M^2 are Kählerian.)

When $n \geqslant 3$, the situation is much more complicated. Calabi [Ca3] proved that there is a non-Kählerian structure on $X \times T^2_{\mathbf{C}}$, where X is a hyperelliptic curve with genus $g = 2k + 1$, $k \geqslant 0$. On the other hand, we know that the only Kählerian structures on $X \times T^2_{\mathbf{C}}$ is the standard one.

Are there non-Kählerian complex structures on compact locally irreducible Hermitian symmetric spaces which are covered by bounded domains?

Yau made the following conjecture: Suppose $M^n (n>2)$ is a compact Kähler manifold with negative sectional curvature; then there exist a unique Kählerian complex structure. This statement is false if the condition "negative sectional curvature" is replaced by "negative bisectional curvature".

For a locally Hermitian symmetric space M^n, Calabi and Vesentini [CV] proved that $H^1(TM) = 0$ when $n \geqslant 2$. Siu [S2] partially settled Yau's conjecture by proving the following theorem: If M^n is a compact Kähler manifold with strongly negative curvature, then the Kähler structure on M is unique.

Now suppose that M is Kähler and diffeomorphic to a compact quotient D^n/Γ of the unit ball $D \subset \mathbf{C}^n$. Prior to Siu's theorem, Yau [Y1] proved that the Kähler structure on M is unique by using the following Chern number inequality:

$$(-1)^n \cdot c_1^{n-2} \cdot c_2 \geqslant \frac{(-1)^n n}{2(n+1)} \cdot c_1^n, \tag{2}$$

where $c_1(M) < 0$. The question is: When is the complex structure on M unique? This is not known for $n \geqslant 3$. The only known result is that every complex structure on M is hyperbolic in the sense of Kobayashi, i.e., there are no non-constant holomorphic maps from $\mathbf{C}$ to M.

Inequality (2) also gives the uniqueness of the Kähler structure on $\mathbf{CP}^n$. For n odd this result is due to Hirzebruch and Kodaira [HK]. We remark that in these kinds of rigidity problems, harmonic maps seem to be very useful. In particular, modifications of Siu's $\partial\bar{\partial}$-Bochner-Kodaira would hopefully be useful (see Siu [S2] and Sampson [Sa]).

For the deformation of Kähler structures to algebraic structures, we have the well-known Kodaira conjecture: Every compact Kähler manifold can be deformed to an algebraic manifold. This is known when $n = 2$; in fact, Kodaira [Ko] proved that every compact Kähler surface can be deformed to an algebraic surface. The Kodaira conjecture is not known for $n \geqslant 3$. In particular, if M^n is a non-algebraic compact Kähler manifold and TM is its holomorphic tangent bundle, is $H^1(TM) \neq 0$? Since a compact Kähler manifold with $h^{2,0} = 0$ is algebraic, a related question is: If M is Kähler, does $h^{2,0} \neq 0$ imply $H^1(TM) \neq 0$? (It is easy to construct a map from $H^{2,0}(M)$ to $H^1(T(M))$.)

3. Uniformization

In the one complex dimensional case, we know that every Riemannian surface is one of the following:

$\mathbf{CP}^1$: the Riemannian sphere, which has a unique complex structure,

E: an elliptic curve, which is covered holomorphically by $\mathbf{C}$,

$\Sigma_g, (g > 1)$: a surface covered holomorphically by the unit disk $D \subseteq \mathbf{C}$.

In higher dimensions, many results and classifications come from trying to generalize the above classification. One wants to know under what geometric conditions is M biholomorphic to a higher dimensional analogue of $\mathbf{CP}^1$, E or $\Sigma_g, (g > 1)$. This corresponds to the manifold being elliptic, parabolic or hyperbolic. As is usual, uniqueness will be in the sense of biregular, birational or unirational. In the non-compact case, one basically tries to tame infinity and compactify M as a Zariski open set of some projective algebraic variety $\bar{M}$ so that $M = \bar{M}\backslash\{\text{subvariety}\}$.

A. Elliptic manifolds

Frankel [Fr] conjectured that any compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{CP}^n$; he proved this when $n = 2$. Later, Mori [Mo1] and Siu-Yau [SY1] proved the general case independently. In fact, Mori proved the Hartshorne conjecture under the weaker assumption that M has an ample tangent bundle.

The following is conjectured in [Y6]. If M is a simply connected compact Kähler manifold with nonnegative bisectional curvature, then M is isometric to a product of Hermitian symmetric spaces and complex projective spaces (not necessarily with Fubini-Study metric).

S. Bando [B1] proved this when $n = 3$. Mok and Zhong [MZ] proved that if, in addition, M is Einstein then M is biholomorphically isometric to a Hermitian symmetric space.

Recently, H.-D. Cao and B. Chow [CC] proved the conjecture assuming in addition M has nonnegative curvature operator. Even more recently, Mok claimed to prove the complete conjecture.

Let M^n be a compact Kähler manifold with positive Ricci curvature (this equivalent to $c_1(M) > 0$). We have the following questions:

(1) Under what condition is M^n unirational? Namely, does there exist a rational map from $\mathbf{CP}^n$ to M^n?

(2) Are there only a finite number (in the topological sense) of n-dimensional algebraic manifolds with positive first Chern class?

(3) Is it true that $c_1(M)^n$ is bounded by a constant depending only on n?

For $n = 2$, M^2 is a del Pezzo surface and (1), (2) and (3) are true. For $n = 3$. M^3 is a Fano 3-fold, i.e., an algebraic 3-manifold with ample anticanonical bundle. Mori and Mukai [MM] give a complete classification of Fano 3-folds with second Betti number $b_2(M) \geqslant 2$. In fact, they proved that there are exactly 87 types of Fano 3-folds with $b_2(M) \geqslant 2$, up to deformation; moreover, a Fano 3-fold with $6 \leqslant b_2(M) \leqslant 10$ is isomorphic to $\mathbf{CP}^1 \times S_{11-b_2(M)}$ where S_d denotes the del Pezzo surface of degree d. The Fano 3-folds with $b_2 = 1$ are called Fano 3-folds of the first kind and were classified by Isokovskih [Is]. Using the above classification, questions (2) and (3) are easily checked to be true, but question (1) is not completely known even for $n = 3$. Using certain properties of conic fiber spaces over $\mathbf{CP}^2$, one can prove that some types of Feno's 3-folds, such as cubic 3-folds in $\mathbf{CP}^4$ are unirational. One does not know if every quartic 3-fold in $\mathbf{CP}^4$ is unirational; see the survey by Beauville [Be] for further details. By the way, before the classification of Mori and Mukai, S. M. L'vovskiĭ [Lv] proved that $c_1(M)^3 \leqslant c_1(\mathbf{CP}^3) = 64$ for Fano 3-folds by Riemann-Roch theorem and a detailed study of families of rational curves C with $(-K_M, C) = 4$. It is interesting to study the families of rational curves in Fano manifolds. Finally, for $n \geqslant 4$, the validity of (1), (2) and (3) are not known. Mori-Mukai recently proved M is uniruled. One more problem is if M^n is rationally connected. It is not hard to see that rational connectedness is stronger than uniruledness, but weaker than unirationalness.

Recall that Gromov's theorem [Gr] says that there is a constant $c(n)$ depending only on n such that $\sum_{i=0}^{n} b_i(M^n) \leqslant c(n)$ for any Riemannian manifold M^n with nonnegative sectional curvature. When M is Kähler, can one replace the condition "nonnegative sectional curvature" by "positive Ricci curvature"? One would also like to understand algebraic manifolds with Kodaira dimension $K(M) = -\infty$, i.e., $H^0(M, K^m) = 0$ for each $m > 0$, where K denotes the canonical line bundle. When $n = 2$, they are either rational surfaces or ruled surfaces.

B. *Parabolic manifolds*

Suppose M^n is a compact Kähler manifold which can be holomorphically covered by $\mathbf{C}^n$. Is it true that M^n can be also covered by the complex torus $T^n_{\mathbf{C}}$? For $n = 2$, Iitaka [Ii] proved that this is true. When $n \geqslant 3$, it is not known. Even in the case $n = 2$, the Kähler condition cannot be dropped (otherwise there exist counterexamples).

Let M^n be a noncompact complete Kähler manifold with positive sectional curvature; is M biholomorphic to $\mathbf{C}^n$? This question has been open for a long time. Siu-Yau [SY2] and Mok-Siu-Yau [MSY] proved the following. Let M be a complete noncompact Kähler manifold, $p \in M$ and $r(x) = \operatorname{dist}(x, p)$. Then

(a) If $\pi_1(M) = 0$, $-A/r^{2+\varepsilon} \leqslant K_M \leqslant 0$ for some $\varepsilon > 0$, then M is biholomorphic isometric to $\mathbf{C}^n$.

(b) If $|K_M| \leqslant A(1/r^2)^{1+\varepsilon}$ and A small enough, then M^n is biholomorphic to $\mathbf{C}^n$. If in addition $K_M < 0$, then M^n is isometric to $\mathbf{C}^n$ with the flat metric.

(c) If $K_M \geqslant 0$, $0 \leqslant R \leqslant A/r^{2+\varepsilon}$ and $\operatorname{vol}\big(B(p, r)\big) \geqslant Cr^{2n}$, then M is biholomorphic to $\mathbf{C}^n$.

Here A and C are any positive constants; K_M and R denote the sectional and scalar curvatures of M, respectively.

Mok [Mk1] improved these results by weakening the bound $1/r^{2+\varepsilon}$ to $1/r^2$. More precisely, he proved the following:

(d) If M has positive bisectional curvature, $0 < R \leqslant A/r^2$ and $\operatorname{vol}\big(B(p, r)\big) \geqslant Cr^{2n}$ for some positive constants A and C, then M is biholomorphic to an affine algebraic variety X.

Let M^n be an algebraic manifold with Kodaira dimension $K(M) = 0$, i.e., there exists $m_0 > 0$ such that $\dim H^0(M, K^{m_0}) > 0$, and for all $m \geqslant 0$, $\dim H^0(M, K^m) \leqslant C$ for some C independent of m, where K denotes the canonical line bundle. Can one classify these manifolds? Note that $c_1(M) = 0$ is a special case of $K(M) = 0$. When $n = 2$, there are exactly two classes of algebraic manifolds with Kodaira dimension $K(M) = 0$, quotients of abelian varieties or $K-3$ surfaces. For $n \geqslant 3$, this is unknown; the case $n = 3$ would be important for physics in view of the superstring theory. It is not known how to classify the topology of threefolds with $c_1 = 0$. Are there only finite number of such manifolds? Do they always admit rational curves if $\pi_1(M) = 0$?

C. Hyperbolic manifolds

If M^n is an algebraic manifold with negative sectional curvature, can M be holomorphically (branched) covered by a bounded domain $\Omega \subseteq \mathbf{C}^n$?

A weaker question is: If $\tilde{M}$ is a simply connected Kähler manifold with negative sectional curvature, are there enough bounded holomorphic functions on $\tilde{M}$ to separate points and give local coordinates? So far, no non-constant holomorphic functions have been proved to exist on $\tilde{M}$ even under the assumption that $\tilde{M}$ covers a compact manifold M.

B. Wong [Wo2] proved that if $\Omega \subseteq \mathbf{C}^n$ is a bounded domain with smooth boundary and Ω covers a compact manifold, then Ω is the ball. P. Yang [Yg] proved that if Ω is a bounded symmetric domain in $\mathbf{C}^n$ with rank greater than one, then there does not exist any Kähler metric on Ω with holomorphic bisectional curvature bounded between two negative constants. In particular, Ω cannot cover any compact Kähler manifold with negative bisectional curvature. Hence if a bounded domain Ω covers a compact Kähler manifold with negative curvature, it must be rather nonsmooth.

Recently, Mostow and Siu [MS] constructed a Kähler surface M^2 with negative sectional curvature by delicately piecing together the Poincaré metric of the 2-ball with the Bergman metric of the domain $\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbf{C}^n$. They proved that the universal cover $\tilde{M}$ of M is not the ball by showing that the Chern numbers of $\tilde{M}$ satisfy $c_1^2 < 3c_2$. This manifold is not diffeomorphic to a locally symmetric space and it is not known whether the universal cover is a bounded domain. Is it possible that a complete non-compact Kähler manifold with (topologically) trivial tangent bundle which covers a compact algebraic manifold is in fact biholomorphic to a domain?

For algebraic surfaces with positive canonical line bundle, does $|c_2/c_1^2 - 1/3|$ small enough imply that M has a Kähler metric with negative sectional curvature? This is not known.

The topology of algebraic surfaces is a very important subject. By the recent activity of Freedman and Donaldson, it seems reasonable to believe that every simply connected four-dimensional smooth manifold can be written as a connected sum of algebraic surfaces (possibly with different orientation). Very strong conclusions on the irreducibility of simply connected algebraic surfaces was recently asserted by Donaldson. Apparently only $\mathbf{CP}^2$ factors can occur if one wants to write it as a connected sum of differentiable manifolds. Perhaps simply connected four-dimensional manifolds with such irreducible condition is diffeomorphic to an algebraic surface.

It is more difficult to predict the topology of algebraic surfaces when the fundamental group is not finite. Shafarevich did make the conjecture that universal cover of any algebraic manifold is holomorphically convex. This may give some information about the topology besides the known inequality on Chern numbers.

4. Analytic objects

In order to understand the complex structure, it is important to understand the analytic objects attached to the structure. Here we give two examples:

A. *Holomorphic maps and vector bundles*

For a complex manifold M, the natural holomorphic vector bundles associated to it are TM, TM^*, $\Lambda^k TM$, $\otimes^k TM$, etc. Of special importance is the canonical line bundle $K = \Lambda^n TM^*$.

By blowing up points or submanifolds, one can get additional analytic objects. The Riemann-Roch theorem, which relates a topological invariant to an analytic invariant, is an important tool in constructing analytic objects or invariants from the given topological or analytic information.

The Yang-Mills theory is often useful in constructing holomorphic vector bundles and other objects over Kähler manifolds. Taubes [T1] used the anti-self-dual solutions to the Yang-Mills equations to construct holomorphic vector bundles of rank two over Kähler surfaces M^2. Is it possible to use this theory to recover the author's theorem that if M^2 is simply connected and its cup product is positive definite, then M^2 is biholomorphic to $\mathbf{CP}^2$?

Taubes [T2] also constructed holomorphic vector bundles over Kähler surfaces under the assumption of an inequality between the two Chern numbers (see also Donaldson [D1] and [D2]). So far, the above arguments only work in the two dimensional case. For higher dimensions, there is no good way to construct holomorphic vector bundles. The idea of Taubes can be extended to construct holomorphic vector bundles over high dimensional manifold. But it is not clear how large a class can one achieve in such a way.

B. *Analytic cycles*

Recall that by an analytic cycle, one simply means the formal sum of analytic subvarieties. Let M^n be an algebraic manifold and $V \subseteq M$ an analytic subvariety of codimension p. Then the fundamental cohomology class η_V of V belongs to $H^{p,p}(M) \cap H^{2p}(M; \mathbf{Z})$. Recall that an element

$\alpha \in H^{2p}(M; \mathbf{Q})$ is analytic if it can be represented by a linear combination, with rational coefficients, of the fundamental classes of subvarieties of codimension p, i.e., $\alpha = \sum_{i=1}^{k} b_i \eta_{V_i}$, where $b_i \in \mathbf{Q}$ and V_i is a subvariety of M. Clearly, every analytic element in $H^{2p}(M; \mathbf{Q})$ belongs to $H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$. Conversely, we have the Hodge Conjecture: Every element $\alpha \in H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$ is analytic. This is true when $p = 1$ and is called the Lefschetz theorem on (1, 1)-classes; it is not known for $p \geqslant 2$.

In a Kähler manifold M^n every analytic subvariety is area-minimizing. This follows in a straightforward way from the formulae of Wirtinger and Stokes. Conversely, under suitable conditions, area-minimizing submanifolds become subvarieties. For example, Siu-Yau [SY1] proved that if $f: \mathbf{CP}^1 \to M^n$ is energy-minimizing and the bisectional curvature of M is positive, then f is either holomorphic or anti-holomorphic.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M)$ group of projective transformations of $\mathbf{CP}^N$. Set

$$B(X, X) = \frac{d^2}{dt^2} (\mathrm{Vol}\, g_t(M)) \bigg|_{t=0}, \quad \text{where} \quad X = \frac{dg_t}{dt}.$$

They proved that the trace of B is negative unless M is a subvariety.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M) \cap H^{2p}(M; \mathbf{Q})$, define a volume function as follows: $\mathrm{Vol}: PGL(N+1, \mathbf{C}) \to \mathbf{R}$ where $\mathrm{Vol}: g \to \inf_C \{\mathrm{Vol}_g(C) \mid C \text{ represents } \alpha\}$. Here α is the Poincaré dual of β and $\mathrm{Vol}_g(C)$ is the volume with respect to the metric $(g \circ f)^* ds_0^2$ where ds_0^2 denotes the Fubini-Study metric on $\mathbf{CP}^N$. If there exists a holomorphic C representing α, then $\mathrm{Vol}_g(\alpha) = \mathrm{Vol}_g(C)$ is independent of the choice of g. Hence Vol is a constant function which attains its minimum. On the other hand, if Vol has a minimum, then Lawson-Simons' argument shows that there exists a holomorphic C representing α. Therefore the Hodge conjecture would be proved if one could show the minimum of Vol is attained.

Siu [S2] obtained the following result. Let M be a compact Kähler manifold with strongly negative curvature. Then any element in $H_{2k}(M; \mathbf{Z})$, for $k \geqslant 2$, can be represented by an analytic subvariety if it can be represented by the continuous image of a compact Kähler manifold. His argument used the Bochner type formula for $\bar{\partial} f \wedge \partial \bar{f}$ to get the complex analyticity of the harmonic map f. However, it seems to be difficult to decide which cycles can be represented by continuous images of Kähler manifolds.

§ 6. Canonical Metrics Over Complex Manifolds

Given a complex manifold M, one could like to find "canonical" metrics on M so that one can produce invariants for the complex structure. One natural requirement for canonical metrics is that the totality of them can be parametrized by a finite dimension space and that they be invariant under the group of biholomorphisms.

1. The Bergman, Kobayashi-Royden and Caratheodory metrics

The Bergman metric was first introduced as a natural metric defined on bounded domains in $\mathbf{C}^n$. Later, the definition was generalized to complex manifolds whose canonical bundle K admit sufficiently many sections. For a domain D in $\mathbf{C}^n$, let $H^2(D)$ denote the space of square integrable holomorphic functions of D. Choose an orthonormal basis $\{\phi_i\}$ of this space. Then the Bergman kernel is defined as

$$K(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w)\,.$$

Notice that the definition of the Bergman kernel is independent of the choice of orthonormal basis. Moreover, K is holomorphic in the variables z and $\bar{w}$.

We can now define the Bergman metric by

$$ds^2 = \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)\, dz_i \otimes d\bar{z}_j\,.$$

The naturality of the Bergman metric can easily be seen from the definition of the Bergman kernel. Let D_1 and D_2 be two domains in $\mathbf{C}^n$, and $K_1(z, w)$ and $K_2(z', w')$ their respective Bergman kernels. If $F: D_1 \to D_2$ is a biholomorphism, then K_1 and K_2 are related by the formula

$$K_1(z, w) = K_2(f(z), f(w)) \det\left(\frac{\partial F}{\partial z}\right) \overline{\det\left(\frac{\partial F}{\partial w}\right)}\,.$$