§4. Minimal submanifolds

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cists. The simplest symmetric spaces are the real and complex projective spaces. In [Ca1], Calabi gave an effective parametrization of isotropic harmonic maps from surfaces into real projective space. Following Calabi and the work of physicists, Eells and Wood [EW2] set up a bijective correspondence between full isotropic harmonic maps $\phi : M^2 \to \mathbf{CP}^n$ and pairs (f, r) where $f : M^2 \to \mathbf{CP}^n$ is a full holomorphic map and $0 \leqslant r \leqslant n$ is an integer (see [Ca1] and [EW2] for definitions). Their idea is based on the fact that if $\phi : M \to \mathbf{CP}^n$ is a full isotropic map, then for some $r, s, r + s = n$, the map

$$f = [(\phi \oplus D'' \phi \oplus \cdots \oplus (D'')^{r-1} \phi \oplus (D' \phi \oplus \cdots \oplus (D')^s \phi)]^{\perp}$$

is full holomorphic. Here D' and D'' are the $(1, 0)$ and $(0, 1)$ components of the covariant derivative.

Later, Bryant ([Br1], [Br2]) treated conformal harmonic maps from surfaces into S^6 and S^4. Inspired by the twistor construction of Calabi and Penrose, he considered a restricted class of conformal harmonic maps, namely superminimal surfaces. (Note that Hopf already studied these surfaces in its primitive form). He established a one-to-one correspondence between superminimal surfaces and curves horizontal in $\mathbf{CP}^3$ with respect to the twistor fibration $\mathbf{CP}^3 \xrightarrow{T} S^4$. By constructing such a curve, Bryant showed that any Riemann surface be conformally immersed as a minimal surface in S^4. For the construction in a general 4-manifold, see [ESa].

Recently, K. Uhlenbeck [U3] has dealt with the space H of harmonic maps from a simply-connected 2-dimensional domain into a real Lie group $G_{\mathbf{R}}$ (which is the chiral model in the language of theoretical physics). She studied the algebraic structure of the manifold H and its relation with Kac-Moody algebras.

Another uncultivated area in harmonic maps is the classification of harmonic maps from a surface into a Ricci flat Kähler three-fold. The interest in this comes from the study of superstring theory in theoretical physics.

§ 4. Minimal submanifolds

The study of minimal submanifolds is another important topic in differential geometry. In this section we will mainly consider minimal surfaces in compact three manifolds. The minimal surfaces will be assumed to be regular and embedded, except when otherwise indicated.

In general, it is not too difficult to find an immersed minimal surface in a manifold M if the topology of M prevents the surface from contracting to a point. Additionally, one can sometimes prove the embeddedness of this surface. For example, Meeks-Yau [MY] proved that if $\pi_2(M) \neq 0$, then there exist embedded minimal S^2's and $\mathbf{RP}^2$'s in M which span $\pi_2(M)$ as a $\pi_1(M)$-module. This theorem can be used to study finite groups acting on a three dimensional manifold.

It is relatively hard to find minimal surfaces by using the "mountain pass principle". Also, it is also unclear how to apply the Ljusternik-Schnirelmann theory to find many minimal surfaces with restricted topological type. Sacks-Uhlenbeck [Sa-U] used a perturbed energy combined with the "Morse theory" to show that any n-dimensional manifold, with $\pi_k(M) \neq 0$ for some k, contains at least one immersed minimal S^2. This work of Sachs-Uhlenbeck was used by Siu-Yau to settle the Frankel conjecture in Kähler geometry. Recently a similar type of argument was used by M. Micalif [Mc] and D. Moore [MD] to give a proof of the classical pinching theorem in Riemannian geometry. In fact, they need weaker pinching assumptions.

In his thesis, Pitts [Pi] introduced the notion of "almost minimizing varifold", which roughly speaking is a varifold close to a locally minimizing varifold. Using the nontriviality of the homotopy groups of the integral cycle groups [Al], he proved that any manifold of dimension $\leqslant 6$ supports a nonempty, compact, embedded smooth minimal hypersurface. His idea was to apply the mini-max principle to maps from S^1 into integral currents, which are nontrivial under the isomorphism set up by Almgren [Al]. Since this construction is so general, we do not obtain any topological information about the minimal hypersurface. Recently, R. Schoen-L. Simon [SS] generalized Pitts' work. They showed that any manifold admits a minimal hypersurface with the singular set of Hausdorff codimension at least seven.

On certain three manifolds, R. Schoen and the author can give an estimate of the genus of the minimal surface constructed by Pitts. The argument was done a long time ago. Since this has not been published yet, we give an outline of the proof here.

Let M denote the 3-manifold, R, R_{ij} and R_{ijij} its scalar, Ricci and sectional curvature. Let Σ be the minimal surface constructed by Pitts; K, A, and e_3 the Gaussian curvature, second fundamental form and normal vector field of Σ. By Pitts' construction, we know that the minimal surface must have index 1. This condition is equivalent to the nonnegativity of the second eigenvalue of the operator L, where

$$L = -\Delta - \left(\mathrm{Ric}(e_3) + \| A \|^2\right)$$

and Δ is the intrinsic Laplacian of the surface Σ. In other words,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) f^2 \cdot dv \leqslant \int_\Sigma | \nabla f |^2 \cdot dv$$

for all functions f orthogonal to the first eigenfunction u_1.

We are now going to use the concept of conformal area, which is a conformal invariant, to give an upper estimate of the second eigenvalue λ_2 in terms of M and the genus of Σ.

Let $F : \Sigma \to S^n$ be a conformal immersion into the unit n-sphere. Then F composed with any conformal transformation $g \in \mathrm{Conf}(S^n)$ is also a conformal immersion. Since u_1 is a positive function, by using the argument in [LY2], one can find $g_0 \in \mathrm{Conf}(S^n)$ such that $g_0 \circ F \perp u_1$, i.e.,

$$\int_\Sigma (g_0 \circ F) \cdot u_1 dv = 0 .$$

Now consider the new map $g_0 \circ F$, which we will also denote by $F, F = (f_i)$. $\Sigma f_i^2 = 1$. Since Σ has index 1,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) f_i^2 dv \leqslant \int_\Sigma | \nabla f_i |^2 dv$$

and by taking the summation,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) dv \leqslant \int_\Sigma \sum_i | \nabla f_i |^2 \cdot dv .$$

Since F is conformal, $\int_\Sigma \sum_i | \nabla f_i |^2 dv = 2 \,\mathrm{Area}(F(\Sigma))$. Hence

$$2 \inf_F \sup_{g \in \mathrm{Conf}(S^n)} \mathrm{Area}(g \circ F(\Sigma)) \geqslant \int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) dv .$$

The term on the left hand side is a conformal invariant, $V_c(n, \Sigma)$, called the n-dimensional conformal area; its infimum over all n is $V_c(\Sigma)$, the conformal area of the surface Σ.

By using the branched covering of Σ over S^2, one can show $V_c(\Sigma) \leqslant 4\left(\frac{g(\Sigma)}{2}+1\right)\pi$, where $g(\Sigma)$ is the genus of Σ. Hence

$$8\left(\frac{g(\Sigma)}{2}+1\right)\pi \geqslant \int_{\Sigma}(\mathrm{Ric}(e_3)+\|A\|^2)dv\,.$$

On the other hand, since $\mathrm{Ric}(e_3)+\|A\|^2 = \mathrm{Ric}(e_1)+\mathrm{Ric}(e_2)-2K$, if we assume M has nonnegative Ricci curvature, then

$$\int_{\Sigma}(\mathrm{Ric}(e_3)+\|A\|^2) \geqslant -\int_{\Sigma}2\cdot K dv = 4\pi(2g(\Sigma)-2)\,.$$

Combining the previous two inequalities, we have

$$8\pi\left(\frac{g(\Sigma)}{2}+1\right) \geqslant 4\pi(2g(\Sigma)-2)\,.$$

Hence $g(\Sigma) \leqslant 4$, which is the required upper bound for the genus of Σ. Actually, one should be able to improve this estimate since the estimate on $V_c(M)$ is not sharp. Can one generalize the arguments here to study minimal surfaces with higher index?

The reason one would like to estimate the genus of minimal surfaces is because they contain information about the ambient manifold M. For example, an embedded minimal surface in a 3-manifold M with positive scalar curvature provides a good candidate for a Heagard splitting of M. Moreover, if M is a homotopy 3-sphere and the genus of this surface is less than or equal to 2, then M is actually a sphere. Thus, if one can construct a minimal surface which provides a Heagard splitting and find a good bound for its genus, then one has made substantial progress towards the Poincaré conjecture.

An important problem in minimal surface theory is the existence of more than one minimal surface (or even infinitely many) in a manifold. An analogous situation is that of closed geodesics. On a 2-sphere, there exist at least three closed, embedded geodesics. An ellipsoid has exactly three, so this estimate is sharp.

For the three-sphere, one hopes to show that there exist at least four minimal two-spheres. One would also like to know if, for an ellipsoid centered at the origin in $\mathbf{R}^4$, the only minimal 2-spheres are the four coming from the intersections with the coordinate 3-spaces.

Using an idea of Pitts, Smith and Simon [Sm-S] were able to show that any 3-sphere supports an embedded 2-sphere. They considered degree one mappings $F: I \times S^2 \to S^3$ such that on each slice (except the ones at the ends), $F(t, \cdot): S^2 \to S^3$ is an embedding. They showed that by taking the mini-max

$$\min_F \max_{t \in [0,1]} \operatorname{Area}(F(t, S^2)),$$

one obtains an embedded minimal S^2. Can one do similar theorems for homotopic spheres?

Another problem is to understand the space of minimal surfaces with fixed genus g in three-manifolds with positive Ricci curvature. Recently, Choi and Schoen [CS] proved that this space is actually compact for any fixed genus g. We remark that this compactness is new even for the standard sphere. Their proof is based on an upper estimate of the area of a minimal surface Σ_g due to Choi and Wang [CW]. The area bound will then control the convergence of the minimal surfaces. Knowing this compactness theorem, there are still several interesting questions. For example, do there exist continuous families of minimal surfaces when M has no symmetry? When M is symmetric, do all of these continuous families come from the isometry group?

Estimates for the first eigenvalue are always interesting, especially for minimal surfaces. For minimal surfaces in the standard 3-sphere, the coordinate functions are eigenfunctions with eigenvalue 2. The author conjectured that 2 is actually the first eigenvalue in this case. In an attempt to prove the conjecture, Choi and A. N. Wang [CW] showed that for a minimal surface in a 3-manifold with Ricci curvature not less than 2, the first eigenvalue λ_1 is at least 1. In terms of the conformal area of the minimal surface, Li and Yau [LY2] obtained the following upper bound for λ_1,

$$\frac{2 \operatorname{conf} \operatorname{area}(\Sigma_g)}{\operatorname{area}(\Sigma_g)} \geqslant \lambda_1(\Sigma_g).$$

It is in this way Choi-Wang obtained an upper bound of the area. It would be interesting to generalize this inequality to higher eigenvalues and also study higher eigenvalues of minimal surfaces.

Let M be a homotopy 3-sphere. If M is not a 3-sphere then it contains a fake 3-disk. Put a metric which is asymptotically a product near the boundary. If we minimize area among all S^2's isotopic to the boundary, then the limiting S^2 will enclose a fake disk. Take a Jordan curve on this S^2 so that it decomposes the S^2 into two regions with equal area. Then one expects this Jordan curve to bound an embedded minimal disk in the fake disk. If one can achieve this, one can shrink the S^2 more and obtain a contradiction which will give a proof of the Poincaré conjecture.

In conclusion, minimal surface theory is surprisingly successful in being applied to three dimensional topology. I believe that a more thorough study of minimal surfaces will reveal more secrets about three manifolds.

§ 5. Kähler Geometry

In the following we consider four basic topics in complex geometry.

1. Existence of complex and almost complex structure.
2. Existence of Kähler and algebraic structures on complex manifolds.
3. Uniformization problems and the parametrization of metrics.
4. Analytic objects over complex manifolds, e.g., analytic cycles, holomorphic vector bundles, etc.

We will divide this section into four parts corresponding to these topics.

1. Complex and almost complex structures

Let M be an even dimensional oriented differentiable manifold. The existence of an almost complex structure J is equivalent to a reduction of the structure group of the tangent bundle from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. This is basically an algebraic problem and is well understood.

However, the question of when an almost complex structure is homotopic to an integrable almost complex structure (i.e., one which comes from a complex structure) is much harder. When $n = 1$, every M^2 admits an almost complex structure and every such structure is integrable and algebraic. For $n = 2$, ven de Van [V1] gave several examples of compact M^4's which admit an almost complex structure but not a complex structure. His argument is based on the computations of the first and second Chern classes. When $n \geqslant 3$, there are no such examples known so far. In particular, we do not know whether or not the almost complex manifold S^6 admits a complex structure. This problem has been open for a long time.

The topology of complex surfaces is not well understood. By the works of Donaldson, one may believe that every simply connected four dimensional