3. Rigidity

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Yau [Sc-Y2] have generalized Eells-Sampson's [ES] and Hartman's [Hr] work. They showed that if N is a compact manifold with nonpositive sectional curvature, M is complete and $f : M \to N$ has finite energy, then f is homotopic on compact sets to a harmonic map with finite energy.

Later [Sc-Y3], by explicitly computing the hessian of the distance function d^2 considered as a function on $N \times N$, showed that the set of harmonic maps in a homotopy class is connected (see [Hr] when M is compact) and can be immersed in N as a totally geodesic submanifold. Moreover, it is a point if $\pi_1(N)$ has no nontrivial abelian subgroup and the image of M is neither a point nor a circle. Here we assumed M has finite volume and the harmonic maps have finite energy. (When N is locally symmetric, this is also done by Sunada.) They also applied the theory of harmonic maps to study finite groups acting on a compact manifold.

3. Rigidity

It is natural to ask if harmonic homotopy equivalences are isometries when M and N are both negatively curved Einstein manifolds with dimension $\geqslant 3$. This is based on the uniqueness of harmonic maps into negatively curved manifolds and the Mostow rigidity theorem. If this is true, it would give another proof of the Mostow rigidity theorem in the case of rank one symmetric spaces.

It is a question for negatively curved manifolds M and N, whether a harmonic homotopy equivalence is a diffeomorphism or not. Schoen-Yau [Sc-Y4] and Sampson [Sa] have proved this when M and N are Riemann surfaces. If we only assume non-positivity of curvature, Calabi has constructed a counterexample when N is a torus.

By minimizing the energy among diffeomorphisms, combined with a replacement argument, Jost-Schoen [JS] constructed a harmonic diffeomorphism between surfaces of the same genus without any curvature assumption. (Hence it generalizes a theorem of Schoen-Yau where one assumes the image has non-positive curvature.)

There are plenty of examples of harmonic maps when M and N are Kähler manifolds. In particular, holomorphic maps are harmonic. On the other hand, it was conjectured by Yau that when N has negative curvature, harmonic maps are holomorphic. In attempting to settle this conjecture of Yau, Siu [S2], proved that a harmonic map f is either holomorphic or antiholomorphic provided N is strongly negatively curved and the rank of f is not less than 4 at some point. The assumption of N being strongly negatively curved is similar to the negativity of the curvature operator. One expects to be able to weaken this condition. But, if one only assumes negative bisectional curvature, the analog of Siu's theorem is false. This is because for $M = B^n/\Gamma$ embedded in $\mathbf{CP}^n$ as a regular subvariety, any hyperplane section of M has negative bisectional curvature and it is not rigid in general.

Recently, Jost-Yau [JY1, 2] looked at the complex structure of complex surfaces M homotopy equivalent to $N = D \times D/\Gamma$ where Γ is irreducible. Let $f: M \to N$ be a harmonic homotopy equivalence where M is Kähler. By analyzing the foliation $f^\alpha \equiv$ const., they showed that the universal cover of M is biholomorphic to $D \times D$.

Subsequently, Mok [Mk2] generalized the theorem of Jost-Yau to arbitrary dimension. He also considered the foliation studied by Jost and Yau.

A generalization of the rigidity theorem to quasi-projective manifolds was made by Jost-Yau. They study the complex structure over Hermitian symmetric spaces with finite volume.

For a compact manifold M with strongly nonpositive curvature, one likes to prove M is either locally Hermitian symmetric or that the complex structure is rigid. Sampson [Sa] treated the case where M is Kähler and N is a Riemannian manifold with Hermitian negative curvature, that is $R^N_{ijkl}\, u^i v^j \bar{u}^k \bar{v}^l \leqslant 0$. By applying Bochner's technique in essentially the same way as Siu, he showed that all harmonic maps between M and N are holomorphic. Using Sampson's result, combined with the existence theorem for harmonic maps, we can easily obtain restrictions on the fundamental group of a Kähler manifold.

Another interesting situation is when M and N are Kähler manifolds and N has positive sectional curvature. Is it true that any minimizing harmonic map is holomorphic or antiholomorphic? This is only known when $M = \mathbf{CP}^1$. Also, if we can prove this assuming in addition that N is an irreducible symmetric space, then the conjecture that an irreducible symmetric Kähler manifold has only one Kähler structure is probably true. Notice that for the reducible Kähler manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$, there exists infinitely many complex structures which are Kähler.

4. Harmonic maps in physics

The classification theory of harmonic maps from surfaces to Riemannian manifolds, especially symmetric spaces, is of interest to mathematical physi-