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NONLINEAR ANALYSIS IN GEOMETRY

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The basic purpose of geometry is to give a good description of a class of geometric objects. Usually this means that we have to give a good description of analytic structures over a space and the geometric objects defined by such structures. In many cases, we have to know how to deform these structures and study the dynamics of the geometric objects within these structures. The description of all these geometric phenomena usually are governed by differential equations. As geometric objects are in general curved, most of these equations are nonlinear. Just like in physics, most of the system of equations in geometry are variational in nature. As a matter of fact, almost all the equations that we study in geometry are related to physics. (We take a broad definition of physics here and therefore also include many problems in engineering also.) Perhaps geometry is as real as physics. Historically many problems that were considered by geometers for their own beauty later arose naturally in problems in physics. This often surprised both physicists and geometers. It seems that when nature expresses her own beauty through mathematics, she also uses it to reveal her depth. One of the most recent developments in high energy physics is the superstring theory. It demands a great deal of knowledge from geometry. We expect a continuous joint effort from both physicists and geometers.

Besides physics, geometry is also closely related to topology and algebraic geometry. While topology tells us the very basic nature of space, algebraic geometry provides us numerous natural examples on which we can test our theory. We hope to indicate some of these connections in these lectures.

The lectures will roughly be divided into the following topics:

(I) Linear equations: Spectrum of Laplacian and harmonic functions.

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(II) Semi-linear equations: Yamabe problem related to conformal deformation.

(III) Minimal surface equations and harmonic maps.

(IV) Kähler geometry.

Before we go into those problems in detail, we offer a classification of some views in geometry. In the past, a lot of geometers worked on "local" problems. Nowadays, more attention is paid to "global" problems. It often creates a conflict of views between these two different views of geometers. Actually, as in the theory of differential equations, progress on global problems is based on our understanding of local problems. In fact, some local problems are more difficult than global problems. Let us discuss it in the following:

(i) *Local problem.*

Most local problems in geometry can be reduced via algebra to local existence theorems in differential equations. The algebra that is involved can be very intricate. The work of Griffiths and his coworkers on the local isometric embedding is a good example of this. The Cartan-Kähler theory was devised exactly to accomplish the reduction of geometric problems to local existence theorems like the Cauchy-Kovalevsky theorem. Various implicit function theorems including the Nash-Moser iteration procedure are used in these local problems. When the equations involved are degenerate (i.e., not fully elliptic or hyperbolic or not changing type in a nondegenerate way) the problem of local existence can be extremely difficult. This is especially true when the equations are nonlinear.

An interesting example is the local embedding problem. An old classical problem in geometry is the local metric embedding of a piece of surface into three dimensional Euclidean space. The equation has the form

$$\det\left(u_{ij}-b_{ij}(x, y, \nabla u)\right) = F(x, y, u, \nabla u)\,.$$

When F changes sign or when the zero set of F is complicated, the local existence is very difficult. In fact, Pogovelov and Jacobowitz gave counterexamples when $F \geqslant 0$ and the surface is $C^{2,1}$ ([Pg], [Ja]). Therefore, it was very remarkable when C. S. Lin demonstrated the local existence when the surface is C^{10} ([Ln1, 2]).

In passing, one should also mention the deep work of Kurinishi on local embedding theory on C-R structure.

(ii) *Semi-local theory.*

The basic example is the development of singularities in the involved equations. The understanding of the structure of symmetry is perhaps one of the most difficult subjects in geometry and differential equations. In algebraic geometry, singularity is a much more well-defined concept as "space" here is the zero set of a class of polynomials and singularity is the set of points where locally it does not look like the affine space.

In geometry, singularity is more difficult to define, especially when the structure is governed by the hyperbolic system and the topology of the space is allowed to change. A good example is the singularity developed in general relativity. At this time, we do not really have a good definition of Black Hole which is basically a singularity of the Einstein equations. Until we have a good understanding of the global picture of the Einstein equations with nonsingular initial data, it will be very difficult to give a more precise definition of singularity in general relativity. Notice that in the famous Schwarzchild solution, there is coordinate singularity which can be "cured" by changing the coordinate system. This clearly complicated the problem. Penrose formulated a famous problem called cosmic censorship which gives a prediction for a generic phenomena of how singularities should develop. It is probably the most fundamental problem in classical relativity.

The major problem here is that we have very little understanding of the global behavior of nonlinear hyperbolic systems when the spatial dimension is greater than one. It is almost for sure that a break through will be accomplished in geometry if we know this type of equation better. For the Einstein system, the best work was recently achieved by D. Christodoulou who gave a very good understanding of the spherical symmetric case ([Cr]). One expects that a lot of problems concerning spherical collapse can be understood through his work.

For nonlinear elliptic problems, one has a well developed regularity theory. The works go back to Bernstein, Schauder, Morrey, Nirenberg, De Giorgi, Federer, Fleming, Allard, Almgren, Simon, etc. Most of the works are focused on minimal subvarieties. The reason is very simple. Most difficulties arise in nonlinear elliptic equations already appeared in minimal subvarieties. A good understanding in minimal subvarieties usually give a breakthrough for a more general class of nonlinear elliptic equations. Most of the accomplishments in regularity theory of minimal subvariety assume that the minimal subvariety minimizes area in a global way.

The best example is demonstrated by a codimension one area minimizing subvariety. The works of De Giorgi, Federer, Fleming, Almgren, Allard, Simon, and Hardt show that there is no singularity for dimension less than six. Recently, L. Simon gave a very good understanding of isolated singularity by studying a neighborhood of the singularity [Si]. He proved that a neighborhood is always described by the graph of a Hölder continuous function defined on the tangent cone.

(iii) *Global problems.*

A lot of problems related to physics, topology and algebraic geometry are global in nature. Hence most of the works of global geometry are related to these subjects. Roughly speaking, "global" means that we study analytic structure over compact spaces. For non-compact space, we request that the structure be complete in some way. For geometric objects defined by these structures, we would like to know their evolution for all time and their asymptotic behavior.

In many ways, the basic questions are

(1) Given a complete analytic structure, how does one deduce global information from local data?

(2) Given the topology of a space, can we put certain analytic structures over this space?

Hence the second question corresponds to an existence theorem in analysis. The first question corresponds to uniqueness. It should be clear from the statements of these questions that understanding of global topology is essential in the treatment of these problems. It turns out that one can turn the argument around and give new theorems in topology. The most recent example is the spectacular achievement of S. Donaldson ([D3, 4]) of applying gauge field theory to understand four dimensional topology. Here the existence theorems was due to Taubes based on the works of K. Uhlenbeck ([T1, 2], [U1, 2]). Normally existence theorems based on pure information of topology is the first step. Once the analytic structure is established, we can deduce topological consequences from the analytic structure. We hope to be able to give some feelings in the following lectures.

§ 1. Eigenvalues and Harmonic Functions

The most fundamental elliptic operator on a Riemannian manifold M is the Laplace operator Δ. If M is compact then Δ has a discrete spectrum. We denote the spectrum (i.e., the eigenvalues of Δ) by $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots$. It is a basic fact that $\lambda_i \to \infty$ as $i \to \infty$.

There are two basic directions in the study of eigenvalues and they are closely related to each other. The first is the study of the asymptotic behavior of the sequence $\{\lambda_i\}$. The fundamental results can be found in [BGM].

The well-known Weyl's formula gives the first term in the asymptotic expansion for λ_i. It says that

$$\lambda_i \sim C_n i^{\frac{2}{n}} / (\mathrm{vol}\, M)^{\frac{2}{n}} \quad \text{as} \quad i \to \infty ,$$

where C_n is a universal constant depending only on $n = \dim M$.

It is a much harder problem to determine the second term in the asymptotic expansion for λ_i. Ivrii [Iv] has done some significant work in this direction (substantial work was also done by Melrose). His results can be stated briefly as follows. Let M be a compact manifold with boundary $\partial M \neq \emptyset$. We first consider the Dirichlet problem. For a positive real number λ, let $N(\lambda)$ denote the number of eigenvalues (counting multiplicity) which do not exceed λ^2. Under a certain technical assumption related to the set of closed geodesics on M, the following asymptotic formula holds:

$$N(\lambda) = (2\pi)^{-n} W_n(\mathrm{vol}\, M) \cdot \lambda^n - \frac{1}{4}(2\pi)^{-n+1} W_{n-1}(\mathrm{vol}\, \partial M) \cdot \lambda^{n-1} + o(\lambda^{n-1}) .$$

where W_n and W_{n-1} are constants depending only on n. A similar result holds for the Neumann problem. The method Ivrii used was to study the singularity of the fundamental solution of the wave equation $\partial^2 u/\partial t^2 = \Delta u$.

Another problem related to Weyl's formula is the Polyá conjecture. It says the following. If Ω is a bounded domain in $\mathbf{R}^n$, then

$$\lambda_i \geqslant c_n(\mathrm{vol}\, \Omega)^{-2/n}\, i^{2/n}$$

and

$$\mu_i \leqslant C_n(\mathrm{vol}\, \Omega)^{-2/n}\, i^{2/n} .$$

Here $\{\lambda_i\}$ are the eigenvalues for the Dirichlet problem and $\{\mu_i\}$ are the eigenvalues for the Neumann problem.

In [LY1], Li and Yau proved that in the average, the Polyá conjecture is true. The method depends on Fourier transform of the Laplacian. If one can take care of the boundary term for the Fourier transform of high power of the Laplacian, one will be able to settle the conjecture. Furthermore, for any closed manifold M, they also prove

$$\lambda_i \leqslant C_1 + C_2(i+1)^{2/n} \cdot (\mathrm{vol}\, M)^{-2/n} ,$$

where C_1 and C_2 are constants depending only on m, the diameter of M and a lower bound on the Ricci curvature of M.

The heat kernel, or the fundamental solution of the operator $\partial/\partial t - \Delta$, was a basic tool in understanding eigenvalues of the Laplacian. It often gives an estimate of the eigenvalues with less dependence on the geometry. However, except for the first term in the Weyl's asymptotic estimate, the heat kernel argument is not capable to give information for the lower order asymptotic term at this moment. In any case, a lot of information was obtained in the past by studying the trace of the heat kernel which is $\sum_i e^{-\lambda_i t}$. In particular, one can recover the volume, the total scalar curvature, etc. from this infinite series when $t \to 0$. However, since we have to know all the eigenvalues in order to calculate the asymptotic value, it is not an effective way to recover the invariants. Can one find an *effective* way to calculate the heat invariants? When M is a convex domain, one can actually recover the volume. When M is not convex, the problem is not stable and difficult. In any case, in general one cannot recover all the information about the geometry of the manifold (see [Mi]). C. Gordon and Wilson [GW] found a non-trivial continuous family of metrics on a compact manifold with the same spectrum. However, all the known examples of the manifolds with the same spectrum have the property that they are locally isometric to each other. Is this a generic phenomena?

The second direction in the study of the spectrum is to estimate the low eigenvalues, especially λ_1 for a general manifold by using the mini-max principle. An upper bound was found by Cheng [Ch1] depending only on the diameter of the manifold and a lower bound for the Ricci curvature. Later, Li and Yau [LY1] obtained a lower bound for λ_1 depending on the same data. A sharp lower estimate was found by Zhong [Z]. Both of these estimates have nice applications in geometry. Cheng's theorem implies that a compact manifold, whose Ricci curvature is not less than $n - 1$ and whose diameter is π, is isometric to S^n. The estimate of Li-Yau was used by E. Ruh to provide a new proof of a strengthened version of Gromov's theorem on almost flat manifolds.

It is an interesting and important problem to estimate the gap between eigenvalues. For example, one knows that the multiplicity of $\lambda_1(S^2)$ is less than or equal to three (see [Ch2]). Thus one would like to estimate $\lambda_4 - \lambda_1$. For a convex domain Ω in $\mathbf{R}^n$, $\lambda_1 < \lambda_2$ for the Dirichlet boundary condition. In [SWYY], I. Singer, B. Wong, S. S.-T. Yau and S.-T. Yau gave a lower bound for $\lambda_2 - \lambda_1$ for convex domains. The basic idea of the proof is as follows. Let f_1 and f_2 denote the first two eigenfunctions. Since Ω is convex, the function $u = f_1/f_2$ is well-defined and smooth up to the boundary of Ω. Let $G = |\nabla u|^2 + \lambda(\mu - u)^2$ where $\lambda = \lambda_2 - \lambda_1$ and $\mu = \sup_\Omega u$ by the maximum principle, it is not hard to see that

$$G \leqslant \sup_{\partial\Omega} G = \sup_{\partial\Omega} \lambda(\mu - u)^2 .$$

This implies

$$|\nabla u|^2 + \lambda(\sup u - u)^2 \leqslant \lambda[(\sup u - \inf u)^2 - (\sup u - u)^2]$$

and hence

$$\sqrt{\lambda} \geqslant \frac{|\nabla u|}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}} .$$

Integrating this inequality along the line segment joining two points where the minimum and maximum of u are obtained, we obtain

$$\lambda_2 - \lambda_1 = \lambda \geqslant \frac{\pi^2}{4d^2} ,$$

where d is the diameter of Ω. Can one improve the constant of this inequality so that equality actually holds for the interval?

It should be clear that the understanding of eigenvalues depends crucially on understanding the eigenfunctions. A basic part of the eigenfunction is its zero set. It is called the nodal set. Even for two dimensional manifolds, we do not really understand the nodal set. A very famous old problem was to study the nodal line of the second eigenfunction of a convex domain. It was conjectured that it cannot enclose any compact subset of the domain. Recently C. S. Lin [Ln3] proved it under the assumption that the domain has a symmetry. Another interesting question can be posed as follows. Let l_n be the length of the n-th eigenfunction. Can we obtain an asymptotic estimates of l_n? It looks like that it has order $\sqrt{\lambda_n}$ where λ_n is the n-th eigenvalue. The difficult question is to give an upper estimate of l_n.

In the following we will consider the case where M is a complete noncompact manifold. In this case, there should be two theories. An L^2- and an L^∞-theory. We first consider the L^2-theory. The spectrum is then not discrete in general. However, one may still ask: When does $-\Delta$ have an eigenvalue? That is, there exists $f \in L^2(M)$, $f \neq 0$, such that $\Delta f = -\lambda f (\lambda > 0)$. We hope that the following are true.

(1) M doesn't have a pure point spectrum when M is complete with $K \geqslant 0$ (K is the sectional curvature). Escobar [Es] demonstrated this when M is rotationally symmetric outside a compact set.

(2) M does not have an infinite number of eigenvalues when M is complete, simply connected with $-C \leqslant K \leqslant -1$.

(3) M has an infinite number of eigenvalues when M is a complete locally symmetric space with finite volume.

The validity of these conjectures are unknown even when $n = \dim M = 2$. It is known that conjecture 3 is true assuming M is the quotient of a symmetric domain by an arithmetic subgroup.

It is especially interesting to understand the case $\lambda = 0$. In particular, when does M have nonconstant harmonic functions with desired properties, and if so, how many are there?

In [Y5], Yau proved that there are no non-constant L^p-harmonic functions on any complete, noncompact manifold for $1 < p < +\infty$. In particular, there are no L^2-harmonic functions besides constants. Because of this, we will concentrate our attention on positive or bounded harmonic functions. The basic problem we would like to discuss here is to give geometric conditions on M so that the Liouville theorem is either true or false. A result due to Yau [Y7] says that if the Ricci curvature is nonnegative, then the only positive harmonic functions are the constants. Such manifolds may be called *strongly parabolic*.

Very recently, P. Li and Tam [LT] investigated the situation where M has nonnegative sectional curvature outside a compact set. He classified all of the bounded or positive harmonic functions on M. It would be nice if one could replace the above condition by nonnegative Ricci curvature. By using Yau's argument, Donnelly showed that the space of positive harmonic functions is finite dimensional. Such manifolds may be called *parabolic*. It will be interesting to prove that manifolds which are uniformly equivalent to a complete manifold with non-negative Ricci curvature are parabolic. In the other direction, one would like to prove that many manifolds are hyperbolic, i.e., non-parabolic. Anderson [A] and Sullivan [Su] were able to solve the Dirichlet problem for simply connected complete manifolds with curvature bounded by two negative constants. Later, Anderson and Schoen [AS] did some beautiful work on positive harmonic functions on these manifolds. They prove the existence of a C^α-homeomorphism between the Martin boundary and the sphere at infinity $S(\infty)$. The identification of the Martin boundary with the sphere at infinity allows us to begin a systematic study of positive harmonic functions. One can prove that every positive harmonic function u on M can be obtained by the following formula,

$$u = \int_{S(\infty)} K(x, Q) d\mu_Q ,$$

where μ is the unique positive Borel measure on $S(\infty)$, and $K(x, Q)$ is the Poisson kernel. This is the Martin representation formula. Anderson and Schoen were able to study the regularity of K.

One can define harmonic measure on the Martin boundary. It is an important question to study the regularity of this measure. Perhaps a lot of classical facts on harmonic measure for bounded domain have analogues here.

It is not known how to carry through the above theory when the curvature is unbounded from below. It is also not known what the Martin boundary looks like when the curvature is only non-positive. For the case of symmetric domain, there is a well-developed theory. It would be nice to be able to understand symmetric domains through this general framework.

Another important question is to prove that a non-compact complete manifold M is hyperbolic if $\lim_{i \to \infty} \lambda_1(\Omega_i) > 0$ where Ω_i is a compact exhaustion of M.

There are many interesting questions concerning harmonic functions on complete manifolds. A function has polynomial growth of degree k if $|f| \leqslant C(1+r)^{k+\varepsilon}$ for any $\varepsilon > 0$, where $r = d(x, p)$ and p is a fixed point on M. A function has linear growth if it satisfies the previous inequality for $k = 1$. One would like to know a bound on the dimension of harmonic functions with linear or polynomial growth on a complete Riemannian manifold with non-negative Ricci curvature. If M is Kähler, the holomorphic functions with polynomial growth form a ring. In this case, one would like to know when this ring is finitely generated, and when the generators may be chosen to have linear growth. This question is very much related to the following conjecture in Kähler geometry. A complete non-compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{C}^n$.

Siu-Yau [S-Y2] and Mok-Siu-Yau [MSY] made an attempt to settle this questions by using the L^2-theory of Hörmander. However, the assumption was rather strong. The method was to construct holomorphic functions with slow growth. Recently, Li-Yau [L-Y3] used arguments from elliptic theory to study linear growth holomorphic functions. They made the assumption that the volume growth of the manifold is polynomial with degree $2n$.

In the other direction, it is a major problem to construct bounded holomorphic functions on a complete simplete connected Kähler manifold with strongly negative curvature. In fact, one would like to prove that it is biholomorphic to a bounded domain in $\mathbf{C}^n$ or at least that bounded holomorphic functions separate points of the manifold. It looks like the problem is very much related to a possible generalization of the classical Corona problem to higher dimensional bounded domains.

§ 2. Yamabe's Equation and Conformally Flat Manifolds

Yamabe's equation is a nonlinear elliptic scalar equation related to the conformal deformation of a metric on a Riemannian manifold. Given a metric g_0 with scalar curvature R_0, let g be a metric pointwise conformal to g_0. Then $g = u^{4/(n-2)}g_0$, where $u > 0$ is a smooth function. The scalar curvature R of g is given by the equation

$$L_0 u = -\gamma_0 \Delta_0 u + R_0 u = R u^{\alpha}, \tag{1}$$

where Δ_0 is the Laplacian with respect to g_0, $\gamma_0 = \dfrac{4(n-1)}{n-2}$, $\alpha = \dfrac{n+2}{n-2}$ and $n = \dim M$.

In [Ya], Yamabe asserted that there is always a solution $u > 0$ to equation (1) with $R = \text{const}$. That is to say, any metric on a compact Riemannian manifold is conformally equivalent to a metric with constant scalar curvature. However, his proof contained an error. This was discovered by Trudinger. Moreover, Trudinger [Tr] showed that (1) could be solved for $R = \text{const.}$ provided the lowest eigenvalue λ_1 of the linear operator L_0 is nonpositive.

Let Y be the functional on $L_1^2(M)$ defined by

$$Y = \int_M (\gamma \,|\, \nabla_0 u \,|^2 + R_0 u^2) \Big/ \left(\int_M R u^{\alpha+1}\right)^{\frac{2}{\alpha+1}}$$

where ∇_0 is the gradient with respect to the metric g_0. By a simple computation, one finds that (1) is the Euler-Lagrange equation for the functional Y.

Aubin [Au1] gave a sufficient condition for Y to have a minimum in $L_1^2(M)$. It can be described as follows. Fix $R \equiv 1$ and let $\sigma(g_0)$ be the minimum of Y, $\Lambda_n = \sigma(\hat{g})$ where $\hat{g}$ is the standard metric on the unit sphere S^n. Then

(a) $\sigma(g_0) \leqslant \Lambda_n$ for any metric g_0,

(b) If $\sigma(g_0) < \Lambda_n$, there exists a smooth function u minimizing Y.

Since u is a solution to (1) with R = const., Yamabe's conjecture translates to whether or not $\sigma(g_0) < \Lambda_n$ for metrics not conformal to the standard metric on S^n. Aubin [Au1] proved that if $n \geqslant 6$ and g_0 is not conformally flat, then $\sigma(g_0) < \Lambda_n$. The argument of Aubin is local. He constructed a function supported in a small open set which is radial. Thus, the remaining cases are when $n = 3, 4$ or 5 and when M is locally conformally flat for $n \geqslant 6$.

Recently, R. Schoen [Sc] gave a complete solution to Yamabe's conjecture. His argument is global and uses the generalized positive mass theorem ([ScY5]), Schoen gave a higher order estimate for $Y(u^\varepsilon)$ for a suitable sequence $\{u^\varepsilon\}$ in the case where M is conformally flat or $n = 3$. The case $n \geqslant 4$ requires perturbation arguments using again the positive mass theorem.

We may also consider the same questions for complete, noncompact manifolds. Recently, Schoen announced some new results. A particularly interesting result is as follows. If M has the topological type of $S^n - \{p_1, \ldots, p_k\}$ for $k > 1$, then one can find a metric with scalar curvature equal to one in each conformal class of complete metrics.

Another topic related to the Yamabe conjecture is the study of (locally) conformally flat manifolds. A theorem of Kuiper [Ku] says that for any conformally flat, simply connected manifold M, one can find an open conformal mapping from M into the standard sphere which is unique up to a conformal diffeomorphism of S^n. This map is called the developing map. We denote its image by Ω and let $\Lambda = S^n - \Omega$.

Schoen and Yau [ScY6] obtained results relating the Hausdorff dimension of Λ to the sign of the scalar curvature of M. The results can be stated as follows.

1. If M is a complete (possibly compact) conformally flat manifold with positive scalar curvature $R \geqslant 1$, then the developing map is a conformal diffeomorphism into S^n. Hence, conformally, M is covered by an open subset of S^n. The argument here uses crucially the Green's function of the conformal operator.

2. If M is a compact conformally flat manifold with positive scalar curvature, then $\mu_{\frac{n}{2}-1}(\Lambda) = 0$. Here μ_k is the k dimensional Hausdorff measure.

3. If M is a compact Riemannian manifold covered conformally by $\Omega \subset S^n$ with $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$, then M admits a metric with scalar curvature $R \geqslant 0$ in the same conformal class. It is conjectured that if $R \geqslant 0$, then $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$.

The basic idea is that by using the developing map, we can reduce the problems to the study of a scalar equation, namely the Yamabe equation on an open subset of S^n. The remaining parts of the proofs are relatively easy. By using the same technique, Schoen and Yau proved that for a compact conformally flat manifold with positive scalar curvature, $\pi_i(M) = 0$ for $2 \leqslant i \leqslant n/2$. Some of their results are also valid for complete manifolds.

§ 3. Harmonic Maps

Harmonic maps are important objects in geometry and analysis. They appear naturally as critical points of an energy functional of the appropriate function space. Harmonic maps reflect a lot about the geometric properties of manifolds.

Given Riemannian manifolds M and N, consider the mapping space $C^r(M, N)$. One problem is to find nice (i.e., canonical) representatives in this space. For a map $f: M \to N$ we define its energy by $E(f) = \int_M |df|^2 dV_M$. A harmonic map is a critical point of this energy. The first question is that of existence, uniqueness and regularity.

1. Existence, uniqueness and regularity

The first major work was done by J. Eells and L. Sampson [ES]. They proved the existence of a harmonic map in any homotopy class in the case where M and N are compact manifolds with $K_N \leqslant 0$. They deformed an arbitrary map through a nonlinear heat equation. By passing to the limit, with the appropriate estimates, one obtains a harmonic map in this way. In fact, harmonic maps are unique in their homotopy classes if $K_N < 0$ and rank $\geqslant 2$ [Hr]. Later, R. Hamilton [Ha] using the same method as in [ES] together with delicate estimates, settled the Dirichlet problem when M

is a manifold with boundary. This type of argument breaks down when we drop the non-positivity condition. For example Eells and Wood [EW1] have shown that there does not exist a degree 1 map from a 2-torus to a 2-sphere.

Instead of looking for harmonic maps in a homotopy class, one can look for harmonic maps with the same action on π_1. We say that two maps $f, g: M \to N$ are π_1- equivalent if $f_* = g_*: \pi_1(M) \to \pi_1(N)$. When M is a Riemann surface, L. Lemaire [Lm] proved the existence of a regular, energy minimizing harmonic map in the class of π_1-equivalent maps.

Another treatment of this problem was given by Sacks-Uhlenbeck [SaU] and R. Schoen-S. T. Yau [Sc-Y1]. Schoen-Yau considered the function space L_1^2 and showed that for $u \in L_1^2(M, N)$, u_* is well-defined and preserved under the weak limit. Using the class $\{f \in L_1^2(M, N) \mid f_* = (f_0)_*\}$ which is weakly closed, combined with the regularity of minimizing harmonic maps from a surface, one can show the existence of a smooth harmonic map in this class.

Schoen-Yau's argument could be generalized to higher dimensions by restricting the map f to the two skeleton of M. (This was also observed by White [Wh].) It is reasonable to expect that one can produce an energy minimizing harmonic map whose action on $\pi_2(M)$ has some resemblance to a given map.

For minimizing harmonic maps, R. Schoen and K. Uhlenbeck [ScU1, 2] have done fundamental work. By delicate use of comparison maps, they showed that the Hausdorff dimension of the singular set of energy minimizing harmonic maps is of codimension at least three. Their theorem can be used to recover the former theorems of Eells-Sampson and Sachs-Uhlenbeck.

2. Noncompact manifolds

The theory for harmonic maps between noncompact manifolds is more complicated than when the manifolds are compact. One reason is that when we choose a minimizing sequence of maps, their energies may not be concentrated in a bounded region. On the other hand, one hopes that this can be prevented by making suitable topological assumptions on the manifolds.

For L^2-harmonic maps, i.e., weakly harmonic maps with finite energy, one can sometimes prove existence by making geometric or topological restrictions. When N is a manifold with nonpositive curvature, Schoen and

Yau [Sc-Y2] have generalized Eells-Sampson's [ES] and Hartman's [Hr] work. They showed that if N is a compact manifold with nonpositive sectional curvature, M is complete and $f : M \to N$ has finite energy, then f is homotopic on compact sets to a harmonic map with finite energy.

Later [Sc-Y3], by explicitly computing the hessian of the distance function d^2 considered as a function on $N \times N$, showed that the set of harmonic maps in a homotopy class is connected (see [Hr] when M is compact) and can be immersed in N as a totally geodesic submanifold. Moreover, it is a point if $\pi_1(N)$ has no nontrivial abelian subgroup and the image of M is neither a point nor a circle. Here we assumed M has finite volume and the harmonic maps have finite energy. (When N is locally symmetric, this is also done by Sunada.) They also applied the theory of harmonic maps to study finite groups acting on a compact manifold.

3. Rigidity

It is natural to ask if harmonic homotopy equivalences are isometries when M and N are both negatively curved Einstein manifolds with dimension $\geqslant 3$. This is based on the uniqueness of harmonic maps into negatively curved manifolds and the Mostow rigidity theorem. If this is true, it would give another proof of the Mostow rigidity theorem in the case of rank one symmetric spaces.

It is a question for negatively curved manifolds M and N, whether a harmonic homotopy equivalence is a diffeomorphism or not. Schoen-Yau [Sc-Y4] and Sampson [Sa] have proved this when M and N are Riemann surfaces. If we only assume non-positivity of curvature, Calabi has constructed a counterexample when N is a torus.

By minimizing the energy among diffeomorphisms, combined with a replacement argument, Jost-Schoen [JS] constructed a harmonic diffeomorphism between surfaces of the same genus without any curvature assumption. (Hence it generalizes a theorem of Schoen-Yau where one assumes the image has non-positive curvature.)

There are plenty of examples of harmonic maps when M and N are Kähler manifolds. In particular, holomorphic maps are harmonic. On the other hand, it was conjectured by Yau that when N has negative curvature, harmonic maps are holomorphic. In attempting to settle this conjecture of Yau, Siu [S2], proved that a harmonic map f is either holomorphic or antiholomorphic provided N is strongly negatively curved and the rank of f is not less than 4 at some point. The assumption of N being strongly negatively curved is similar to the negativity of the curvature operator. One expects to be able to weaken this condition. But, if one only assumes negative bisectional curvature, the analog of Siu's theorem is false. This is because for $M = B^n/\Gamma$ embedded in $\mathbf{CP}^n$ as a regular subvariety, any hyperplane section of M has negative bisectional curvature and it is not rigid in general.

Recently, Jost-Yau [JY1, 2] looked at the complex structure of complex surfaces M homotopy equivalent to $N = D \times D/\Gamma$ where Γ is irreducible. Let $f: M \to N$ be a harmonic homotopy equivalence where M is Kähler. By analyzing the foliation $f^\alpha \equiv$ const., they showed that the universal cover of M is biholomorphic to $D \times D$.

Subsequently, Mok [Mk2] generalized the theorem of Jost-Yau to arbitrary dimension. He also considered the foliation studied by Jost and Yau.

A generalization of the rigidity theorem to quasi-projective manifolds was made by Jost-Yau. They study the complex structure over Hermitian symmetric spaces with finite volume.

For a compact manifold M with strongly nonpositive curvature, one likes to prove M is either locally Hermitian symmetric or that the complex structure is rigid. Sampson [Sa] treated the case where M is Kähler and N is a Riemannian manifold with Hermitian negative curvature, that is $R^N_{ijkl} u^i v^j \bar{u}^k \bar{v}^l \leqslant 0$. By applying Bochner's technique in essentially the same way as Siu, he showed that all harmonic maps between M and N are holomorphic. Using Sampson's result, combined with the existence theorem for harmonic maps, we can easily obtain restrictions on the fundamental group of a Kähler manifold.

Another interesting situation is when M and N are Kähler manifolds and N has positive sectional curvature. Is it true that any minimizing harmonic map is holomorphic or antiholomorphic? This is only known when $M = \mathbf{CP}^1$. Also, if we can prove this assuming in addition that N is an irreducible symmetric space, then the conjecture that an irreducible symmetric Kähler manifold has only one Kähler structure is probably true. Notice that for the reducible Kähler manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$, there exists infinitely many complex structures which are Kähler.

4. Harmonic maps in physics

The classification theory of harmonic maps from surfaces to Riemannian manifolds, especially symmetric spaces, is of interest to mathematical physicists. The simplest symmetric spaces are the real and complex projective spaces. In [Ca1], Calabi gave an effective parametrization of isotropic harmonic maps from surfaces into real projective space. Following Calabi and the work of physicists, Eells and Wood [EW2] set up a bijective correspondence between full isotropic harmonic maps $\phi: M^2 \to \mathbf{CP}^n$ and pairs (f, r) where $f: M^2 \to \mathbf{CP}^n$ is a full holomorphic map and $0 \leqslant r \leqslant n$ is an integer (see [Ca1] and [EW2] for definitions). Their idea is based on the fact that if $\phi: M \to \mathbf{CP}^n$ is a full isotropic map, then for some $r, s, r + s = n$, the map

$$f = [(\phi \oplus D'' \phi \oplus \cdots \oplus (D'')^{r-1} \phi \oplus (D' \phi \oplus \cdots \oplus (D')^s \phi)]^{\perp}$$

is full holomorphic. Here D' and D'' are the $(1, 0)$ and $(0, 1)$ components of the covariant derivative.

Later, Bryant ([Br1], [Br2]) treated conformal harmonic maps from surfaces into S^6 and S^4. Inspired by the twistor construction of Calabi and Penrose, he considered a restricted class of conformal harmonic maps, namely superminimal surfaces. (Note that Hopf already studied these surfaces in its primitive form). He established a one-to-one correspondence between superminimal surfaces and curves horizontal in $\mathbf{CP}^3$ with respect to the twistor fibration $\mathbf{CP}^3 \xrightarrow{T} S^4$. By constructing such a curve, Bryant showed that any Riemann surface be conformally immersed as a minimal surface in S^4. For the construction in a general 4-manifold, see [ESa].

Recently, K. Uhlenbeck [U3] has dealt with the space H of harmonic maps from a simply-connected 2-dimensional domain into a real Lie group $G_{\mathbf{R}}$ (which is the chiral model in the language of theoretical physics). She studied the algebraic structure of the manifold H and its relation with Kac-Moody algebras.

Another uncultivated area in harmonic maps is the classification of harmonic maps from a surface into a Ricci flat Kähler three-fold. The interest in this comes from the study of superstring theory in theoretical physics.

§ 4. MINIMAL SUBMANIFOLDS

The study of minimal submanifolds is another important topic in differential geometry. In this section we will mainly consider minimal surfaces in compact three manifolds. The minimal surfaces will be assumed to be regular and embedded, except when otherwise indicated.

In general, it is not too difficult to find an immersed minimal surface in a manifold M if the topology of M prevents the surface from contracting to a point. Additionally, one can sometimes prove the embeddedness of this surface. For example, Meeks-Yau [MY] proved that if $\pi_2(M) \neq 0$, then there exist embedded minimal S^2's and $\mathbf{RP}^2$'s in M which span $\pi_2(M)$ as a $\pi_1(M)$-module. This theorem can be used to study finite groups acting on a three dimensional manifold.

It is relatively hard to find minimal surfaces by using the "mountain pass principle". Also, it is also unclear how to apply the Ljusternik-Schnirelmann theory to find many minimal surfaces with restricted topological type. Sacks-Uhlenbeck [Sa-U] used a perturbed energy combined with the "Morse theory" to show that any n-dimensional manifold, with $\pi_k(M) \neq 0$ for some k, contains at least one immersed minimal S^2. This work of Sachs-Uhlenbeck was used by Siu-Yau to settle the Frankel conjecture in Kähler geometry. Recently a similar type of argument was used by M. Micalif [Mc] and D. Moore [MD] to give a proof of the classical pinching theorem in Riemannian geometry. In fact, they need weaker pinching assumptions.

In his thesis, Pitts [Pi] introduced the notion of "almost minimizing varifold", which roughly speaking is a varifold close to a locally minimizing varifold. Using the nontriviality of the homotopy groups of the integral cycle groups [Al], he proved that any manifold of dimension $\leqslant 6$ supports a nonempty, compact, embedded smooth minimal hypersurface. His idea was to apply the mini-max principle to maps from S^1 into integral currents, which are nontrivial under the isomorphism set up by Almgren [Al]. Since this construction is so general, we do not obtain any topological information about the minimal hypersurface. Recently, R. Schoen-L. Simon [SS] generalized Pitts' work. They showed that any manifold admits a minimal hypersurface with the singular set of Hausdorff codimension at least seven.

On certain three manifolds, R. Schoen and the author can give an estimate of the genus of the minimal surface constructed by Pitts. The argument was done a long time ago. Since this has not been published yet, we give an outline of the proof here.

Let M denote the 3-manifold, R, R_{ij} and R_{ijij} its scalar, Ricci and sectional curvature. Let Σ be the minimal surface constructed by Pitts; K, A, and e_3 the Gaussian curvature, second fundamental form and normal vector field of Σ. By Pitts' construction, we know that the minimal surface must have index 1. This condition is equivalent to the nonnegativity of the second eigenvalue of the operator L, where

$$L = -\Delta - (\mathrm{Ric}(e_3) + \| A \|^2)$$

and Δ is the intrinsic Laplacian of the surface Σ. In other words,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) f^2 \cdot dv \leqslant \int_\Sigma | \nabla f |^2 \cdot dv$$

for all functions f orthogonal to the first eigenfunction u_1.

We are now going to use the concept of conformal area, which is a conformal invariant, to give an upper estimate of the second eigenvalue λ_2 in terms of M and the genus of Σ.

Let $F: \Sigma \to S^n$ be a conformal immersion into the unit n-sphere. Then F composed with any conformal transformation $g \in \mathrm{Conf}(S^n)$ is also a conformal immersion. Since u_1 is a positive function, by using the argument in [LY2], one can find $g_0 \in \mathrm{Conf}(S^n)$ such that $g_0 \circ F \perp u_1$, i.e.,

$$\int_\Sigma (g_0 \circ F) \cdot u_1 dv = 0 .$$

Now consider the new map $g_0 \circ F$, which we will also denote by F, $F = (f_i)$. $\Sigma f_i^2 = 1$. Since Σ has index 1,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) f_i^2 dv \leqslant \int_\Sigma | \nabla f_i |^2 dv$$

and by taking the summation,

$$\int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) dv \leqslant \int_\Sigma \sum_i | \nabla f_i |^2 \cdot dv .$$

Since F is conformal, $\int_\Sigma \sum_i | \nabla f_i |^2 dv = 2 \,\mathrm{Area}(F(\Sigma))$. Hence

$$2 \inf_F \sup_{g \in \mathrm{Conf}(S^n)} \mathrm{Area}(g \circ F(\Sigma)) \geqslant \int_\Sigma (\mathrm{Ric}(e_3) + \| A \|^2) dv .$$

The term on the left hand side is a conformal invariant, $V_c(n, \Sigma)$, called the n-dimensional conformal area; its infimum over all n is $V_c(\Sigma)$, the conformal area of the surface Σ.

By using the branched covering of Σ over S^2, one can show $V_c(\Sigma) \leqslant 4\left(\frac{g(\Sigma)}{2}+1\right)\pi$, where $g(\Sigma)$ is the genus of Σ. Hence

$$8\left(\frac{g(\Sigma)}{2}+1\right)\pi \geqslant \int_{\Sigma}(\mathrm{Ric}(e_3)+\|A\|^2)dv\,.$$

On the other hand, since $\mathrm{Ric}(e_3)+\|A\|^2=\mathrm{Ric}(e_1)+\mathrm{Ric}(e_2)-2K$, if we assume M has nonnegative Ricci curvature, then

$$\int_{\Sigma}(\mathrm{Ric}(e_3)+\|A\|^2) \geqslant -\int_{\Sigma}2\cdot K dv = 4\pi(2g(\Sigma)-2)\,.$$

Combining the previous two inequalities, we have

$$8\pi\left(\frac{g(\Sigma)}{2}+1\right) \geqslant 4\pi(2g(\Sigma)-2)\,.$$

Hence $g(\Sigma) \leqslant 4$, which is the required upper bound for the genus of Σ. Actually, one should be able to improve this estimate since the estimate on $V_c(M)$ is not sharp. Can one generalize the arguments here to study minimal surfaces with higher index?

The reason one would like to estimate the genus of minimal surfaces is because they contain information about the ambient manifold M. For example, an embedded minimal surface in a 3-manifold M with positive scalar curvature provides a good candidate for a Heagard splitting of M. Moreover, if M is a homotopy 3-sphere and the genus of this surface is less than or equal to 2, then M is actually a sphere. Thus, if one can construct a minimal surface which provides a Heagard splitting and find a good bound for its genus, then one has made substantial progress towards the Poincaré conjecture.

An important problem in minimal surface theory is the existence of more than one minimal surface (or even infinitely many) in a manifold. An analogous situation is that of closed geodesics. On a 2-sphere, there exist at least three closed, embedded geodesics. An ellipsoid has exactly three, so this estimate is sharp.

For the three-sphere, one hopes to show that there exist at least four minimal two-spheres. One would also like to know if, for an ellipsoid centered at the origin in $\mathbf{R}^4$, the only minimal 2-spheres are the four coming from the intersections with the coordinate 3-spaces.

Using an idea of Pitts, Smith and Simon [Sm-S] were able to show that any 3-sphere supports an embedded 2-sphere. They considered degree one mappings $F: I \times S^2 \to S^3$ such that on each slice (except the ones at the ends), $F(t, \cdot): S^2 \to S^3$ is an embedding. They showed that by taking the mini-max

$$\min_F \max_{t \in [0,1]} \operatorname{Area}(F(t, S^2)),$$

one obtains an embedded minimal S^2. Can one do similar theorems for homotopic spheres?

Another problem is to understand the space of minimal surfaces with fixed genus g in three-manifolds with positive Ricci curvature. Recently, Choi and Schoen [CS] proved that this space is actually compact for any fixed genus g. We remark that this compactness is new even for the standard sphere. Their proof is based on an upper estimate of the area of a minimal surface Σ_g due to Choi and Wang [CW]. The area bound will then control the convergence of the minimal surfaces. Knowing this compactness theorem, there are still several interesting questions. For example, do there exist continuous families of minimal surfaces when M has no symmetry? When M is symmetric, do all of these continuous families come from the isometry group?

Estimates for the first eigenvalue are always interesting, especially for minimal surfaces. For minimal surfaces in the standard 3-sphere, the coordinate functions are eigenfunctions with eigenvalue 2. The author conjectured that 2 is actually the first eigenvalue in this case. In an attempt to prove the conjecture, Choi and A. N. Wang [CW] showed that for a minimal surface in a 3-manifold with Ricci curvature not less than 2, the first eigenvalue λ_1 is at least 1. In terms of the conformal area of the minimal surface, Li and Yau [LY2] obtained the following upper bound for λ_1,

$$\frac{2 \operatorname{conf} \operatorname{area}(\Sigma_g)}{\operatorname{area}(\Sigma_g)} \geqslant \lambda_1(\Sigma_g).$$

It is in this way Choi-Wang obtained an upper bound of the area. It would be interesting to generalize this inequality to higher eigenvalues and also study higher eigenvalues of minimal surfaces.

Let M be a homotopy 3-sphere. If M is not a 3-sphere then it contains a fake 3-disk. Put a metric which is asymptotically a product near the boundary. If we minimize area among all S^2's isotopic to the boundary, then the limiting S^2 will enclose a fake disk. Take a Jordan curve on this S^2 so that it decomposes the S^2 into two regions with equal area. Then one expects this Jordan curve to bound an embedded minimal disk in the fake disk. If one can achieve this, one can shrink the S^2 more and obtain a contradiction which will give a proof of the Poincaré conjecture.

In conclusion, minimal surface theory is surprisingly successful in being applied to three dimensional topology. I believe that a more thorough study of minimal surfaces will reveal more secrets about three manifolds.

§ 5. Kähler Geometry

In the following we consider four basic topics in complex geometry.

1. Existence of complex and almost complex structure.
2. Existence of Kähler and algebraic structures on complex manifolds.
3. Uniformization problems and the parametrization of metrics.
4. Analytic objects over complex manifolds, e.g., analytic cycles, holomorphic vector bundles, etc.

We will divide this section into four parts corresponding to these topics.

1. Complex and almost complex structures

Let M be an even dimensional oriented differentiable manifold. The existence of an almost complex structure J is equivalent to a reduction of the structure group of the tangent bundle from $GL(2n, \mathbf{R})$ to $GL(n, \mathbf{C})$. This is basically an algebraic problem and is well understood.

However, the question of when an almost complex structure is homotopic to an integrable almost complex structure (i.e., one which comes from a complex structure) is much harder. When $n = 1$, every M^2 admits an almost complex structure and every such structure is integrable and algebraic. For $n = 2$, ven de Van [V1] gave several examples of compact M^4's which admit an almost complex structure but not a complex structure. His argument is based on the computations of the first and second Chern classes. When $n \geqslant 3$, there are no such examples known so far. In particular, we do not know whether or not the almost complex manifold S^6 admits a complex structure. This problem has been open for a long time.

The topology of complex surfaces is not well understood. By the works of Donaldson, one may believe that every simply connected four dimensional compact manifold is the connected sum of algebraic surfaces. For nonsimply connected algebraic surfaces, it is more difficult to speculate. The basic problem is to find a way to construct complex structures. Perhaps one can ask the following question. Suppose M is a compact almost complex manifold satisfying $\chi(M) = 3\tau(M)$ and covered topologically by $\mathbf{R}^4$. (Here $\chi(M)$ is the Euler number and $\tau(M)$ is the index of M.) If every abelian subgroup of $\pi_1(M)$ is infinite cyclic, does M admit a complex structure so that M is covered holomorphically by the unit ball in $\mathbf{C}^2$? The Lefschetz theorem may be useful in the above question.

2. Kähler and algebraic structures

Let M^n be an n complex dimensional compact manifold with complex structure J. The first question is: When is J Kählerian, i.e., (M, J) admits a Kähler metric? Harvey-Lawson [H-L] gave an intrinsic characterization of the Kählerian condition if and only if M carries no positive currents which are the (1, 1)-components of boundaries. Hodge theory gives a lot of necessary conditions for complex manifolds to be Kähler. In particular, their even Betti numbers must be positive and their odd Betti numbers are even. Also, when (M, J) is Kählerian, its rational homotopy type is determined by its rational cohomology, see Deligne-Griffiths-Morgan-Sullivan [DGMS].

Now suppose M is a Kähler manifold, i.e., M has some Kählerian complex structure. When does M admit a non-Kählerian complex structure? When does M have a unique complex (or Kählerian) structure?

When $n = 2$, every compact complex surface with even first Betti number is Kählerian. (This follows from the classification of Kodaira because Miyaoka [M1] and Siu [S1] proved respectively that elliptic surfaces with even first Betti number and $K-3$ surfaces are Kählerian. From this one concludes that among the seven classes of surfaces in Kodaira's classification, the first five are Kählerian for every complex structure. The remaining two classes of surfaces have odd first Betti number and hence admit no Kähler metrics. In particular, one sees that on a Kähler surface M^2, all complex structures on M^2 are Kählerian.)

When $n \geqslant 3$, the situation is much more complicated. Calabi [Ca3] proved that there is a non-Kählerian structure on $X \times T^2_{\mathbf{C}}$, where X is a hyperelliptic curve with genus $g = 2k + 1$, $k \geqslant 0$. On the other hand, we know that the only Kählerian structures on $X \times T^2_{\mathbf{C}}$ is the standard one.

Are there non-Kählerian complex structures on compact locally irreducible Hermitian symmetric spaces which are covered by bounded domains?

Yau made the following conjecture: Suppose $M^n (n > 2)$ is a compact Kähler manifold with negative sectional curvature; then there exist a unique Kählerian complex structure. This statement is false if the condition "negative sectional curvature" is replaced by "negative bisectional curvature".

For a locally Hermitian symmetric space M^n, Calabi and Vesentini [CV] proved that $H^1(TM) = 0$ when $n \geqslant 2$. Siu [S2] partially settled Yau's conjecture by proving the following theorem: If M^n is a compact Kähler manifold with strongly negative curvature, then the Kähler structure on M is unique.

Now suppose that M is Kähler and diffeomorphic to a compact quotient D^n/Γ of the unit ball $D \subset \mathbf{C}^n$. Prior to Siu's theorem, Yau [Y1] proved that the Kähler structure on M is unique by using the following Chern number inequality:

$$(2) \qquad (-1)^n \cdot c_1^{n-2} \cdot c_2 \geqslant \frac{(-1)^n n}{2(n+1)} \cdot c_1^n,$$

where $c_1(M) < 0$. The question is: When is the complex structure on M unique? This is not known for $n \geqslant 3$. The only known result is that every complex structure on M is hyperbolic in the sense of Kobayashi, i.e., there are no non-constant holomorphic maps from $\mathbf{C}$ to M.

Inequality (2) also gives the uniqueness of the Kähler structure on $\mathbf{CP}^n$. For n odd this result is due to Hirzebruch and Kodaira [HK]. We remark that in these kinds of rigidity problems, harmonic maps seem to be very useful. In particular, modifications of Siu's $\partial\bar{\partial}$-Bochner-Kodaira would hopefully be useful (see Siu [S2] and Sampson [Sa]).

For the deformation of Kähler structures to algebraic structures, we have the well-known Kodaira conjecture: Every compact Kähler manifold can be deformed to an algebraic manifold. This is known when $n = 2$; in fact, Kodaira [Ko] proved that every compact Kähler surface can be deformed to an algebraic surface. The Kodaira conjecture is not known for $n \geqslant 3$. In particular, if M^n is a non-algebraic compact Kähler manifold and TM is its holomorphic tangent bundle, is $H^1(TM) \neq 0$? Since a compact Kähler manifold with $h^{2,0} = 0$ is algebraic, a related question is: If M is Kähler, does $h^{2,0} \neq 0$ imply $H^1(TM) \neq 0$? (It is easy to construct a map from $H^{2,0}(M)$ to $H^1(T(M))$.)

3. Uniformization

In the one complex dimensional case, we know that every Riemannian surface is one of the following:

$\mathbf{CP}^1$: the Riemannian sphere, which has a unique complex structure,

E: an elliptic curve, which is covered holomorphically by $\mathbf{C}$,

Σ_g, $(g > 1)$: a surface covered holomorphically by the unit disk $D \subseteq \mathbf{C}$.

In higher dimensions, many results and classifications come from trying to generalize the above classification. One wants to know under what geometric conditions is M biholomorphic to a higher dimensional analogue of $\mathbf{CP}^1$, E or Σ_g, $(g > 1)$. This corresponds to the manifold being elliptic, parabolic or hyperbolic. As is usual, uniqueness will be in the sense of biregular, birational or unirational. In the non-compact case, one basically tries to tame infinity and compactify M as a Zariski open set of some projective algebraic variety $\bar{M}$ so that $M = \bar{M} \backslash \{\text{subvariety}\}$.

A. *Elliptic manifolds*

Frankel [Fr] conjectured that any compact Kähler manifold with positive bisectional curvature is biholomorphic to $\mathbf{CP}^n$; he proved this when $n = 2$. Later, Mori [Mo1] and Siu-Yau [SY1] proved the general case independently. In fact, Mori proved the Hartshorne conjecture under the weaker assumption that M has an ample tangent bundle.

The following is conjectured in [Y6]. If M is a simply connected compact Kähler manifold with nonnegative bisectional curvature, then M is isometric to a product of Hermitian symmetric spaces and complex projective spaces (not necessarily with Fubini-Study metric).

S. Bando [B1] proved this when $n = 3$. Mok and Zhong [MZ] proved that if, in addition, M is Einstein then M is biholomorphically isometric to a Hermitian symmetric space.

Recently, H.-D. Cao and B. Chow [CC] proved the conjecture assuming in addition M has nonnegative curvature operator. Even more recently, Mok claimed to prove the complete conjecture.

Let M^n be a compact Kähler manifold with positive Ricci curvature (this equivalent to $c_1(M) > 0$). We have the following questions:

(1) Under what condition is M^n unirational? Namely, does there exist a rational map from $\mathbf{CP}^n$ to M^n?

(2) Are there only a finite number (in the topological sense) of n-dimensional algebraic manifolds with positive first Chern class?

(3) Is it true that $c_1(M)^n$ is bounded by a constant depending only on n?

For $n = 2$, M^2 is a del Pezzo surface and (1), (2) and (3) are true. For $n = 3$. M^3 is a Fano 3-fold, i.e., an algebraic 3-manifold with ample anti-canonical bundle. Mori and Mukai [MM] give a complete classification of Fano 3-folds with second Betti number $b_2(M) \geqslant 2$. In fact, they proved that there are exactly 87 types of Fano 3-folds with $b_2(M) \geqslant 2$, up to deformation; moreover, a Fano 3-fold with $6 \leqslant b_2(M) \leqslant 10$ is isomorphic to $\mathbf{CP}^1 \times S_{11-b_2(M)}$ where S_d denotes the del Pezzo surface of degree d. The Fano 3-folds with $b_2 = 1$ are called Fano 3-folds of the first kind and were classified by Isokovskih [Is]. Using the above classification, questions (2) and (3) are easily checked to be true, but question (1) is not completely known even for $n = 3$. Using certain properties of conic fiber spaces over $\mathbf{CP}^2$, one can prove that some types of Feno's 3-folds, such as cubic 3-folds in $\mathbf{CP}^4$ are unirational. One does not know if every quartic 3-fold in $\mathbf{CP}^4$ is unirational; see the survey by Beauville [Be] for further details. By the way, before the classification of Mori and Mukai, S. M. L'vovskiĭ [Lv] proved that $c_1(M)^3 \leqslant c_1(\mathbf{CP}^3) = 64$ for Fano 3-folds by Riemann-Roch theorem and a detailed study of families of rational curves C with $(-K_M, C) = 4$. It is interesting to study the families of rational curves in Fano manifolds. Finally, for $n \geqslant 4$, the validity of (1), (2) and (3) are not known. Mori-Mukai recently proved M is uniruled. One more problem is if M^n is rationally connected. It is not hard to see that rational connectedness is stronger than uniruledness, but weaker than unirationalness.

Recall that Gromov's theorem [Gr] says that there is a constant $c(n)$ depending only on n such that $\sum_{i=0}^{n} b_i(M^n) \leqslant c(n)$ for any Riemannian manifold M^n with nonnegative sectional curvature. When M is Kähler, can one replace the condition "nonnegative sectional curvature" by "positive Ricci curvature"? One would also like to understand algebraic manifolds with Kodaira dimension $K(M) = -\infty$, i.e., $H^0(M, K^m) = 0$ for each $m > 0$, where K denotes the canonical line bundle. When $n = 2$, they are either rational surfaces or ruled surfaces.

B. *Parabolic manifolds*

Suppose M^n is a compact Kähler manifold which can be holomorphically covered by $\mathbf{C}^n$. Is it true that M^n can be also covered by the complex torus $T^n_{\mathbf{C}}$? For $n = 2$, Iitaka [Ii] proved that this is true. When $n \geqslant 3$, it is not known. Even in the case $n = 2$, the Kähler condition cannot be dropped (otherwise there exist counterexamples).

Let M^n be a noncompact complete Kähler manifold with positive sectional curvature; is M biholomorphic to $\mathbf{C}^n$? This question has been open for a long time. Siu-Yau [SY2] and Mok-Siu-Yau [MSY] proved the following. Let M be a complete noncompact Kähler manifold, $p \in M$ and $r(x) = \operatorname{dist}(x, p)$. Then

(a) If $\pi_1(M) = 0$, $-A/r^{2+\varepsilon} \leqslant K_M \leqslant 0$ for some $\varepsilon > 0$, then M is biholomorphic isometric to $\mathbf{C}^n$.

(b) If $|K_M| \leqslant A(1/r^2)^{1+\varepsilon}$ and A small enough, then M^n is biholomorphic to $\mathbf{C}^n$. If in addition $K_M < 0$, then M^n is isometric to $\mathbf{C}^n$ with the flat metric.

(c) If $K_M \geqslant 0$, $0 \leqslant R \leqslant A/r^{2+\varepsilon}$ and $\operatorname{vol}(B(p, r)) \geqslant Cr^{2n}$, then M is biholomorphic to $\mathbf{C}^n$.

Here A and C are any positive constants; K_M and R denote the sectional and scalar curvatures of M, respectively.

Mok [Mk1] improved these results by weakening the bound $1/r^{2+\varepsilon}$ to $1/r^2$. More precisely, he proved the following:

(d) If M has positive bisectional curvature, $0 < R \leqslant A/r^2$ and $\operatorname{vol}(B(p, r)) \geqslant Cr^{2n}$ for some positive constants A and C, then M is biholomorphic to an affine algebraic variety X.

Let M^n be an algebraic manifold with Kodaira dimension $K(M) = 0$, i.e., there exists $m_0 > 0$ such that $\dim H^0(M, K^{m_0}) > 0$, and for all $m \geqslant 0$, $\dim H^0(M, K^m) \leqslant C$ for some C independent of m, where K denotes the canonical line bundle. Can one classify these manifolds? Note that $c_1(M) = 0$ is a special case of $K(M) = 0$. When $n = 2$, there are exactly two classes of algebraic manifolds with Kodaira dimension $K(M) = 0$, quotients of abelian varieties or $K-3$ surfaces. For $n \geqslant 3$, this is unknown; the case $n = 3$ would be important for physics in view of the superstring theory. It is not known how to classify the topology of threefolds with $c_1 = 0$. Are there only finite number of such manifolds? Do they always admit rational curves if $\pi_1(M) = 0$?

C. *Hyperbolic manifolds*

If M^n is an algebraic manifold with negative sectional curvature, can M be holomorphically (branched) covered by a bounded domain $\Omega \subseteq \mathbf{C}^n$?

A weaker question is: If $\tilde{M}$ is a simply connected Kähler manifold with negative sectional curvature, are there enough bounded holomorphic functions on $\tilde{M}$ to separate points and give local coordinates? So far, no non-constant holomorphic functions have been proved to exist on $\tilde{M}$ even under the assumption that $\tilde{M}$ covers a compact manifold M.

B. Wong [Wo2] proved that if $\Omega \subseteq \mathbf{C}^n$ is a bounded domain with smooth boundary and Ω covers a compact manifold, then Ω is the ball. P. Yang [Yg] proved that if Ω is a bounded symmetric domain in $\mathbf{C}^n$ with rank greater than one, then there does not exist any Kähler metric on Ω with holomorphic bisectional curvature bounded between two negative constants. In particular, Ω cannot cover any compact Kähler manifold with negative bisectional curvature. Hence if a bounded domain Ω covers a compact Kähler manifold with negative curvature, it must be rather nonsmooth.

Recently, Mostow and Siu [MS] constructed a Kähler surface M^2 with negative sectional curvature by delicately piecing together the Poincaré metric of the 2-ball with the Bergman metric of the domain $\{(z_1, z_2) \mid |z_1|^2 + |z_2|^2 < 1\}$ in $\mathbf{C}^n$. They proved that the universal cover $\tilde{M}$ of M is not the ball by showing that the Chern numbers of $\tilde{M}$ satisfy $c_1^2 < 3c_2$. This manifold is not diffeomorphic to a locally symmetric space and it is not known whether the universal cover is a bounded domain. Is it possible that a complete non-compact Kähler manifold with (topologically) trivial tangent bundle which covers a compact algebraic manifold is in fact biholomorphic to a domain?

For algebraic surfaces with positive canonical line bundle, does $|c_2/c_1^2 - 1/3|$ small enough imply that M has a Kähler metric with negative sectional curvature? This is not known.

The topology of algebraic surfaces is a very important subject. By the recent activity of Freedman and Donaldson, it seems reasonable to believe that every simply connected four-dimensional smooth manifold can be written as a connected sum of algebraic surfaces (possibly with different orientation). Very strong conclusions on the irreducibility of simply connected algebraic surfaces was recently asserted by Donaldson. Apparently only $\mathbf{CP}^2$ factors can occur if one wants to write it as a connected sum of differentiable manifolds. Perhaps simply connected four-dimensional manifolds with such irreducible condition is diffeomorphic to an algebraic surface.

It is more difficult to predict the topology of algebraic surfaces when the fundamental group is not finite. Shafarevich did make the conjecture that universal cover of any algebraic manifold is holomorphically convex. This may give some information about the topology besides the known inequality on Chern numbers.

4. Analytic objects

In order to understand the complex structure, it is important to understand the analytic objects attached to the structure. Here we give two examples:

A. *Holomorphic maps and vector bundles*

For a complex manifold M, the natural holomorphic vector bundles associated to it are TM, TM^*, $\Lambda^k TM$, $\otimes^k TM$, etc. Of special importance is the canonical line bundle $K = \Lambda^n TM^*$.

By blowing up points or submanifolds, one can get additional analytic objects. The Riemann-Roch theorem, which relates a topological invariant to an analytic invariant, is an important tool in constructing analytic objects or invariants from the given topological or analytic information.

The Yang-Mills theory is often useful in constructing holomorphic vector bundles and other objects over Kähler manifolds. Taubes [T1] used the anti-self-dual solutions to the Yang-Mills equations to construct holomorphic vector bundles of rank two over Kähler surfaces M^2. Is it possible to use this theory to recover the author's theorem that if M^2 is simply connected and its cup product is positive definite, then M^2 is biholomorphic to $\mathbf{CP}^2$?

Taubes [T2] also constructed holomorphic vector bundles over Kähler surfaces under the assumption of an inequality between the two Chern numbers (see also Donaldson [D1] and [D2]). So far, the above arguments only work in the two dimensional case. For higher dimensions, there is no good way to construct holomorphic vector bundles. The idea of Taubes can be extended to construct holomorphic vector bundles over high dimensional manifold. But it is not clear how large a class can one achieve in such a way.

B. *Analytic cycles*

Recall that by an analytic cycle, one simply means the formal sum of analytic subvarieties. Let M^n be an algebraic manifold and $V \subseteq M$ an analytic subvariety of codimension p. Then the fundamental cohomology class η_V of V belongs to $H^{p,p}(M) \cap H^{2p}(M; \mathbf{Z})$. Recall that an element

$\alpha \in H^{2p}(M; \mathbf{Q})$ is analytic if it can be represented by a linear combination, with rational coefficients, of the fundamental classes of subvarieties of codimension p, i.e., $\alpha = \sum_{i=1}^{k} b_i \eta_{V_i}$, where $b_i \in \mathbf{Q}$ and V_i is a subvariety of M. Clearly, every analytic element in $H^{2p}(M; \mathbf{Q})$ belongs to $H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$. Conversely, we have the Hodge Conjecture: Every element $\alpha \in H^{2p}(M; \mathbf{Q}) \cap H^{p,p}(M)$ is analytic. This is true when $p = 1$ and is called the Lefschetz theorem on (1, 1)-classes; it is not known for $p \geqslant 2$.

In a Kähler manifold M^n every analytic subvariety is area-minimizing. This follows in a straightforward way from the formulae of Wirtinger and Stokes. Conversely, under suitable conditions, area-minimizing submanifolds become subvarieties. For example, Siu-Yau [SY1] proved that if $f: \mathbf{CP}^1 \to M^n$ is energy-minimizing and the bisectional curvature of M is positive, then f is either holomorphic or anti-holomorphic.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M)$ group of projective transformations of $\mathbf{CP}^N$. Set

$$B(X, X) = \frac{d^2}{dt^2}(\mathrm{Vol}\, g_t(M))\bigg|_{t=0}, \quad \text{where} \quad X = \frac{dg_t}{dt}.$$

They proved that the trace of B is negative unless M is a subvariety.

Lawson-Simons' argument gives an approach towards the Hodge conjecture. Given an embedding $f: M^n \to \mathbf{CP}^N$ and an element $\beta \in H^{p,p}(M) \cap H^{2p}(M; \mathbf{Q})$, define a volume function as follows: $\mathrm{Vol}: PGL(N+1, \mathbf{C}) \to \mathbf{R}$ where $\mathrm{Vol}: g \to \inf_C \{\mathrm{Vol}_g(C) \mid C \text{ represents } \alpha\}$. Here α is the Poincaré dual of β and $\mathrm{Vol}_g(C)$ is the volume with respect to the metric $(g \circ f)^* ds_0^2$ where ds_0^2 denotes the Fubini-Study metric on $\mathbf{CP}^N$. If there exists a holomorphic C representing α, then $\mathrm{Vol}_g(\alpha) = \mathrm{Vol}_g(C)$ is independent of the choice of g. Hence Vol is a constant function which attains its minimum. On the other hand, if Vol has a minimum, then Lawson-Simons' argument shows that there exists a holomorphic C representing α. Therefore the Hodge conjecture would be proved if one could show the minimum of Vol is attained.

Siu [S2] obtained the following result. Let M be a compact Kähler manifold with strongly negative curvature. Then any element in $H_{2k}(M; \mathbf{Z})$, for $k \geqslant 2$, can be represented by an analytic subvariety if it can be represented by the continuous image of a compact Kähler manifold. His argument used the Bochner type formula for $\bar{\partial} f \wedge \partial \bar{f}$ to get the complex analyticity of the harmonic map f. However, it seems to be difficult to decide which cycles can be represented by continuous images of Kähler manifolds.

§ 6. Canonical Metrics Over Complex Manifolds

Given a complex manifold M, one could like to find "canonical" metrics on M so that one can produce invariants for the complex structure. One natural requirement for canonical metrics is that the totality of them can be parametrized by a finite dimension space and that they be invariant under the group of biholomorphisms.

1. The Bergman, Kobayashi-Royden and Caratheodory metrics

The Bergman metric was first introduced as a natural metric defined on bounded domains in $\mathbf{C}^n$. Later, the definition was generalized to complex manifolds whose canonical bundle K admit sufficiently many sections. For a domain D in $\mathbf{C}^n$, let $H^2(D)$ denote the space of square integrable holomorphic functions of D. Choose an orthonormal basis $\{\phi_i\}$ of this space. Then the Bergman kernel is defined as

$$K(z, w) = \sum_i \phi_i(z)\bar{\phi}_i(w) .$$

Notice that the definition of the Bergman kernel is independent of the choice of orthonormal basis. Moreover, K is holomorphic in the variables z and $\bar{w}$.

We can now define the Bergman metric by

$$ds^2 = \sum \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log K(z, z)\, dz_i \otimes d\bar{z}_j .$$

The naturality of the Bergman metric can easily be seen from the definition of the Bergman kernel. Let D_1 and D_2 be two domains in $\mathbf{C}^n$, and $K_1(z, w)$ and $K_2(z', w')$ their respective Bergman kernels. If $F: D_1 \to D_2$ is a biholomorphism, then K_1 and K_2 are related by the formula

$$K_1(z, w) = K_2\big(f(z), f(w)\big) \det\left(\frac{\partial F}{\partial z}\right) \overline{\det\left(\frac{\partial F}{\partial w}\right)} .$$

If the canonical bundle K of M admits enough global, square integrable sections, we can choose an orthonormal basis $\{\phi_i\}$ of sections which will give rise to an embedding $F: M \to \mathbf{CP}^k$. The pull-back metric $F^*(ds^2)$ is the Bergman metric of M. This definition agrees with the previous definition of the Bergman metric when M is a complex domain because any holomorphic function over D can be though of as a section of K.

Intuitively speaking, a complete understanding of the Bergman metric would give us a clear picture of the geometry of the automorphisms of a domain; it would also provide us with a lot of invariants of the domain. In the past few years there has been a lot of progress based on Fefferman's work [Fe]. Fefferman looked at the asymptotic behavior of $K(z, z)$ near the boundary of a domain. Roughly, he proved that the Bergman kernel has the following expansion along the diagonal.

$$K(z, z) = \phi(z)/\Psi^{n+1}(z) + \tilde{\phi}(z) \log \Psi(z)$$

where $\phi, \tilde{\phi} \in C^\infty(D)$, $\phi|_{\partial D} = 0$, and Ψ is the defining function for the domain D.

Moreover, near the boundary we have

$$K(z, w) = \phi(z, w)/\Psi^{n+1}(z, w) + \tilde{\phi}(z, w) \log \Psi(z, w)$$

where $\phi(z, w)$, $\tilde{\phi}(z, w)$ and $\Psi(z, w)$ are extensions of ϕ, $\tilde{\phi}$ and Ψ, respectively, which satisfy certain conditions.

One would actually like to know more about the boundary behavior of the Bergman kernel and metric, the behavior of the curvature of the metric, and other related geometric properties of the metric when Ω is not smooth. Let Ω be a manifold and ds^2_Ω the Bergman metric. If Ω admits a properly discontinuous group of automorphisms we can consider the quotient manifold Ω/Γ and pull-back its Bergman metric $ds^2_{\Omega/\Gamma}$ to Ω. Kazhdan [Kz] proved that if the discrete automorphism group Γ of Ω has a filtration $\Gamma \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n \supseteq \cdots$ with $[\Gamma_i, \Gamma_{i+1}] < \infty$ and $\bigcap_i \Gamma_i = (1)$, then the pull-backs of the Bergman metrics $ds_i^2 = ds^2_{\Omega/\Gamma_i}$ will converge on Ω to the Bergman metric ds^2_Ω of Ω.

Another interesting direction is to look at the global sections of the powers of the canonical bundle. Consider $H^0(M, K^r)$ for r sufficiently large; a choice of basis gives a map $\phi_r: M \to \mathbf{P}(H^2(M, K^r))$. Taking the $1/r$ multiple of the restriction of Fubini-Study metric of $\mathbf{P}(H^2(M, K^r))$, one has a sequence of metrics on M. One would like to know if, as r tends to infinity, a limiting metric exists. If such a metric does exist, it should be "canonical" and hopefully Kähler-Einstein.

For a complex manifold Ω there are two other intrinsically defined pseudometrics: the Kobayashi-Royden metric and the Caratheodory metric. Let Δ be the Poincaré disk in $\mathbf{C}$. We denote by $\Delta(\Omega)$ the set of holomorphic maps from Ω to Δ, $\Omega(\Delta)$ the set of holomorphic maps from Δ to Ω. Fix the Poincaré distance on Δ. The Caratheodory metric is defined by

$$F_\Omega\colon T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_\Omega(z, z) = \sup\{|f_*(z)| : f \in \Delta(\Omega), f(z) = 0\}.$$

The Kobayashi-Royden metric on Ω is defined by

$$F_k\colon T\Omega \to \mathbf{R}^+ \quad \text{where} \quad F_k(z, \xi) = \inf\{|u| : f \in \Omega(\Delta), f(0) = z, f_*(u) = \xi\}.$$

Clearly, these two intrinsically defined metrics are distance decreasing under holomorphic maps and invariant under biholomorphic maps.

B. Wong [Wo1] has shown that the holomorphic sectional curvature of the Caratheodory metric is less than or equal to -4, whereas the holomorphic sectional curvature of the Kobayashi metric is not less than -4 when the metric is nontrivial (for the Bergman metric, it is known that the holomorphic sectional curvature is not greater than 4). However, one disadvantage of these two metrics is that they are neither bilinear nor smooth on the tangent spaces (F is only upper-semicontinuous in general).

In some special cases we have a better understanding of these two metrics. For example, a manifold with strongly negative holomorphic sectional curvature always admits a nontrivial Kobayashi-Royden metric. The major theorem in this subject is due to Royden who showed that the Kobayashi-Royden metric is actually the Teichmüller metric. It is a curious fact that the Teichmüller metric has constant holomorphic sectional curvature. Can we classify those complex manifolds that admit Finsler metric with constant holomorphic sectional curvature?

Lempert [Le1], [Le2] proved that the Kobayashi and Caratheodory metrics are actually the same for convex domains in $\mathbf{C}^n$. By using the existence of an extremal mapping, he constructed a lot of bounded holomorphic functions. His theory only works for convex domains; still, it is interesting to see how one can generalize his ideas or use these two metrics to construct bounded holomorphic functions on more general manifolds.

Another interesting fact, proved by B. Wong [Wo2], is that if a smooth, bounded domain in $\mathbf{C}^n$ covers a closed manifold, then it must be the unit ball. This partially confirms the conjecture that a bounded convex domain (not required to be smooth) which covers a closed manifold must be symmetric. His proof needed the boundary estimate of the Kobayashi and Caratheodory metrics.

In general, one would like to compare the Bergman, Kobayashi-Royden, Caratheodory metrics and the Kähler-Einstein metric discussed in the next section. We know that the Caratheodory metric is the smallest of the three. This can be seen by using the generalized Schwarz lemma for Kähler manifolds [Y4]. Yau (see the later improvement by Chan-Cheng-Lu) proved that if $f: M \to N$ is a holomorphic map where M is a complete Kähler manifold with Ricci curvature bounded from below by a constant and N is a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant, then f decreases distances up to a constant depending on the curvatures of M and N. Is this true if N is only a Finsler space? If it were true, then one expects that Teichmüller metric is uniformly equivalent to the Kähler-Einstein metric.

2. Kähler-Einstein Metrics on Compact Kähler Manifolds

Let M be a compact Kähler manifold. A necessary condition for the existence of a Kähler-Einstein metric on M is as follows.

(*) There exists a Kähler class Ω such that the first Chern class $c_1(M)$ is cohomologous to some real constant multiple of Ω.

This condition is equivalent to the following:

(*)′ The first Chern class satisfies $c_1(M) > 0$, $c_1(M) = 0$ or $c_1(M) < 0$.

It was proved by the author [Y1], [Y2] that when $c_1(M) = 0$ or $c_1(M) < 0$, (for the latter case see also Aubin [Au3]) there exists in every Kähler class a unique Kähler-Einstein metric. When $c_1(M) > 0$, the space Kähler-Einstein metrics are invariant under automorphism group. However, existence does not hold in general and one would like to impose conditions on M to ensure existence.

We now consider the obstruction, due to Futaki [Fu1], to the existence of Kähler-Einstein metrics when $c_1(M) > 0$; we also consider the notion of "extremal metrics" due to Calabi [Ca2]. Fix a Kähler class $\Omega = [\omega] \in H^{1,1}(M)$ on a compact Kähler manifold M and denote by H_Ω the space of all Kähler metrics with Kähler class Ω. Define the functional

$$F: H_\Omega \to \mathbf{R} \quad \text{by} \quad F: (g) \to \int_M R^2,$$

where R denotes the scalar curvature of the metric g. Calabi called a critical point of this functional an extremal metric. Any Kähler-Einstein metric minimizes $\int_M R^2$ in its Kähler class and hence is an extremal metric.

This follows from the Schwarz inequality and the fact that $\int_M R$ is equal to $c_1(M) \cup \omega^{n-1}$ evaluated on the fundamental class of M, where ω is the Kähler form of g.

Calabi proved that for an extremal metric g, the gradient vector field $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic. He also proved that a decomposition theorem holds, analogous to that of Matsushima and Lichnerowicz for constant scalar curvature, for the automorphism group of M. In particular, he proved that X gives rise to a compact subgroup of Aut (M). Levine [Lv] gave an example of a compact surface M^2 with no compact connected subgroup in Aut (M); hence M^2 does not admit any Kähler-Einstein metrics.

For other examples of when Aut (M) is not reductive, see Sakane [Sk1], [Sk2], Ishikawa-Sukane [I-S] and Yau [Y3]. By the theorems of Calabi or Matsushima-Lichnerowicz, these examples do not admit any Kähler-Einstein metrics. Futaki [Fu1] also has constructed examples where Aut (M) is reductive and we will consider them later. So far, however, all examples of a Kähler manifold with positive first Chern class which does not admit a Kähler-Einstein metric admit nontrivial holomorphic vector field, it is natural to ask the following question: If there exists no nonzero holomorphic vector field on M, and if the tangent bundle of M is stable, can we always minimize the functional F? The motivation for the assumption on the stability will be discussed later. Of course, if the answer to the above question is yes, then (*) would also be a sufficient condition for the existence of Kähler-Einstein metrics.

In fact, suppose $c_1(M) = C[\omega]$ and g is an extremal metric. Since $X = \sum g^{i\bar{j}} \frac{\partial R}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ is holomorphic, it follows that $X = 0$, R is constant and the Ricci form of g is a harmonic form representing $c_1(M)$. One concludes that $R_{i\bar{j}} = Cg_{i\bar{j}}$ from the uniqueness of harmonic forms in a cohomology class; hence g is a Kähler-Einstein metric. Calabi [Ca2] proved that, each extremal metric g is a local, nondegenerate point of the functional F. The metric g also exhibits the greatest possible degree of symmetry compatible with the complex structure of M. Let C_Ω denotes the set of extremal metrics in H_Ω, which is diffeomorphic to a finite dimensional Euclidean space.

Moreover, if one metric in C_Ω has constant scalar curvature, then every metric in C_Ω has constant scalar curvature. One expects that the only critical points of F are global minimums of F, form a connected set, and that the group of automorphisms of M which preserve the class Ω acts transitively on C_Ω.

We now consider Futaki's obstruction to the existence of a Kähler-Einstein metric on compact Kähler manifold M with $c_1(M) > 0$. Let $\eta(M)$ denote the Lie algebra of holomorphic vector fields of M, ω a Kähler form representing $c_1(M)$, and γ_ω its Ricci form which also represents $c_1(M)$. Then $\gamma_\omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial \log\det(g_{ij})$ and hence $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial G$ for some smooth function G. Define the character $f : \eta(M) \to \mathbf{C}$ by $f : X \to \int_M (XG)\cdot\omega^n$. Futaki proved that f is independent of the choice of representative ω of $c_1(M)$. Hence the integer $\delta_M = \dim(\eta(M)/\ker(f))$ depends only on the complex structure of M.

If M has a Kähler-Einstein metric then $\delta_M = 0$; Futaki conjectures that the converse is also true. This would be the case if Calabi's functional F attains a minimum. Since $\gamma_\omega - \omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial G$, one has that $R = n + \Delta G$. Then $f(X) = \int (XG)\omega^n = \int (R^\alpha G_\alpha)\omega^n = \int |\Delta G|^2\omega^n$; hence $\delta_M = 0$ implies that $G =$ constant, i.e., g is a Kähler-Einstein metric.

Using the obstruction δ_M, Futaki gave examples of compact Kähler manifolds with $c_1(M) > 0$, Aut (M) reductive, and $\delta_M = 1$. Hence, there does not exist Kähler-Einstein metrics on these examples. Let H_n denote the hyperplane bundle of $\mathbf{CP}^n$ and $\pi_n : H_n \to \mathbf{CP}^n$ the projection map $(n=1, 2)$. If we let $M^5 = \mathbf{P}(E)$ where $E = \pi_1^*(H_1) + \pi_2^*(H_2)$ is considered as a bundle over $\mathbf{CP}^2$, then M is such an example. The following is the lowest dimensional example. If $H \subseteq \mathbf{CP}^3$ is a hyperplane and $C \subseteq H$ a quadratic curve, then let M be $\mathbf{CP}^3$ blown up along C and at a point outside of H.

Futaki's idea is to construct an obstruction for the Ricci form to be harmonic. For the curvature forms representing the higher Chern classes, see Bando [B2]. For questions related to the character f, see Futaki [Fu2] and Futaki-Morita [F-M]. Bando also proved the uniqueness of Kähler-Einstein metric on M with $c_1(M) > 0$, up to holomorphic automorphisms of M.

3. Hermitian Manifolds and Stable Vector Bundles

We will consider canonical metrics on compact complex manifolds which are not necessarily Kählerian. For Hermitian manifolds in general, it is difficult to find canonical metrics because the Hermitian connection has torsion and hence is not Riemannian. Therefore one would like to assume extra conditions on M. Let g be a Hermitian metric on M and ω its Kähler form. One natural condition is to assume that

$$\partial\bar{\partial}(\omega^{n-1}) = 0\,, \tag{1}$$

which is weaker than the condition of being Kähler. One would like to put more conditions on g, besides (1), to make the metric more canonical. Motivated by the theory of supersymmetry, Hull and Witten [HW] proposed the following condition on ω. Locally one should be able to write ω as $\partial\theta + \bar{\partial}\bar{\theta}$ where θ is a (0, 1) form. Notice that if ω is Kähler, it can always be written as $\partial\bar{\partial}f$.

Let us now demonstrate that the above condition is equivalent to the condition $\partial\bar{\partial}\omega = 0$. Clearly, we have only to prove the condition $\partial\bar{\partial}\omega = 0$ implies that ω can be written in the above form. As $\bar{\partial}\omega$ is a closed form, it is locally exact. By comparting the types, we can find a (0, 2) form Ω and a (1, 1) form ω, so that $\bar{\partial}\omega = \partial\Omega + \bar{\partial}\omega'$ with $\bar{\partial}\Omega = 0$ and $\partial\omega' = 0$. Noticing that $\omega = \bar{\omega}$, we can then prove that $\omega - \omega' - \bar{\omega}' - \Omega - \bar{\Omega}$ is a closed form. Therefore, locally it is exact and we can find a (0, 1) form so that $\omega - \omega' - \bar{\omega}' = \partial\theta + \bar{\partial}\bar{\theta}$. Since $\partial\omega' = 0$, locally ω' is ∂-exact and we have proved locally ω is the form that we seek.

Recently Todorov observed that any compact complex manifold admits a Hermitian form ω with $\partial\bar{\partial}\omega = 0$. Therefore it seems that for any compact complex manifold, it is of interest to study the group obtained by taking the quotient of (1, 1) form ω with $\partial\bar{\partial}\omega = 0$ by the subgroup cosets of $\partial\theta + \bar{\partial}\bar{\theta}$ where θ is globally defined (0, 1) form.

Now let V be a holomorphic vector bundle over a compact manifold M with the property $\partial\bar{\partial}(\omega^{n-1}) = 0$. We can define the degree of the bundle V with respect to ω by

$$\deg_\omega V = \int_M \Xi_1(V) \wedge \omega^{n-1}:$$

where $\Xi_1(V)$ denotes the Ricci form of the bundle V. Since $\partial\bar{\partial}(\omega^{n-1}) = 0$, this definition is independent of the choice of metric on V.

In [U-Y], Uhlenbeck and Yau proved the following:

(2) Suppose V is a holomorphic vector bundle over a compact Kähler manifold M. If V is stable, i.e., $\frac{\deg_\omega V'}{\operatorname{rank} V'} < \frac{\deg_\omega V}{\operatorname{rank} V}$ for every coherent subsheaf $V' \subseteq V$ such that $0 < \operatorname{rank}(V') < \operatorname{rank}(V)$, then there exists a Hermitian-Einstein metric on V which is unique up to a constant.

Conversely, the existence of a Hermitian-Einstein metric on V implies that V is direct sum of stable bundles. This was proved by Kobayashi and Lübke [Lu]. Moreover, it is likely that the condition M be Kähler can be replaced by (1). It should be noted that the above theorem was proved by Donaldson [D2] for algebraic surfaces.

We now state some corollaries of (2). First of all, the symmetric tensor product bundle of a stable holomorphic vector bundle is also stable. Secondly, if V is a stable bundle, then for $r = \operatorname{rank}(V)$,

(3) $$\int_M (2r\, c_2(V) - (r-1)c_1^2(V)) \wedge \omega^{n-2} \geqslant 0,$$

and equality holds if and only if up to finite cover of M, V is a direct sum of line bundles (when $n = 2$, this was due to Bogomolov [Bo]) without dealing with the case of equality. Therefore, if $c_1^2(V) = 0$ then $\int_M c_2(V) \wedge \omega^{n-2} \geqslant 0$ and equality holds if and only if V is flat and unique up to a scalar. These results are in fact generalizations of those in the Riemann surface case. In particular, let V be a holomorphic vector bundle over a Riemann surface Σ_g. Then V is stable and $c_1(V) = 0$ if and only if there exists a Hermitian metric on V with zero curvature, i.e., if and if there is a unitary representation of $\pi_1(\Sigma_g)$ (see Narashimhan and Seshadri [N-S] for details.

We now consider the moduli space of stable vector bundles. Let $M(r, d)$ be a complete family of stable vector bundles of fixed rank r and fixed degree d over a Riemann surface Σ_g. Can one prove that $c_1(M_g) > 0$, in particular, can one construct a Kähler metric on M_g with positive Ricci curvature? Cho [Co] proved that there exists a Kähler metric on $M_g(r, d)$ with nonnegative holomorphic sectional curvature. However, even the positivity of the holomorphic sectional curvature does not imply the positivity of the Ricci curvature. For example, let H be the hyperplane bundle over $\mathbf{CP}^1$ and (1) the trivial line bundle. Then the Hirzebruch surfaces M_d

$= \mathbf{P}(H^d + (1))$ have Kähler metrics with positive holomorphic sectional curvature. On the other hand, for $d \geqslant 3$, M_d does not have positive first Chern class.

4. Chern Number Inequalities

In 1976, the author proved the Calabi conjecture and demonstrated the following Chern number inequality for algebraic manifolds with either ample or trivial canonical line bundles:

$$(*) \qquad (-1)^n c_2 c_1^{n-2} \geqslant \frac{(-1)^n}{2(n+1)} c_1^n$$

where equality holds if and only if M is covered by the ball, i.e., $M = B/\Gamma$ for some $\Gamma \subseteq SU(n, 1)$. Around the same time, Miyaoka [M3], extending the method of Bogomolov, obtained the same inequality for $n = 2$ under the weaker assumption that the Kodaira dimension of the surface is non-negative. However, he has not shown that equality holds if and only if M is covered by the ball.

By studying surfaces with singularities, Cheng and Yau [C-Y2] proved inequality (*) for surfaces of general type (equality holds if and only if M^2 is covered by the ball). The arguments in [C-Y2] can also be generalized to higher dimensions. One can also characterize surfaces M which are biholomorphic to B^n/Γ where $\Gamma \subseteq SU(2, 1)$ is allowed to have fixed points. Note that M is, in general, a variety since Γ may have fixed points.

It is also interesting to study manifolds which satisfy certain Chern number inequalities. Surfaces which satisfy inequality (*) have been studied by Hirzebruch, Deligne, Mostow, etc. A corollary of [Y2] is the following rigidity theorem for Kählerian structures on $\mathbf{CP}^n$: The only Kählerian structure on $\mathbf{CP}^n$ is the standard one; moreover, the only complex structure on $\mathbf{CP}^2$ is the standard one. For n odd, this result was due to Hirzebruch and Kodaira [H-K].

We now sketch the proof of inequality (*) when the canonical line bundle K of M is ample. In this case, there exists a Kähler-Einstein metric on K. For Kähler-Einstein metrics one observes that the Chern integral associated to the left hand side of (*) can be expressed in terms of the length squared of the curvature tensor. Since the Ricci tensor is the only part of the curvature tensor, the right hand side, which can be written as the determinant of the Ricci tensor, can be dominated by the left hand side.

If equality holds for (*), one sees that the integrands of both sides are equal. This last fact turns out to be equivalent to M having constraint holomorphic sectional curvature. Hence equality holds in (*) if and only if M is covered by the ball.

Kähler-Einstein metrics do not exist on algebraic manifolds whose canonical line bundle is not a multiple of some ample line bundle. However, it is still possible to study the inequality (*) for algebraic manifolds whose canonical line bundle is almost ample. In [Y1] it was proven that there exists a Kähler-Einstein metric which is degenerate along the divisor where the canonical line bundle is trivial. Similarly one can require the metric to blow up in a certain way. This fact was used by Cheng and Yau [C-Y2] to prove the inequality (*) for surfaces of general type.

$$(**)\quad c_1(M) \leqslant 0 \text{ on } M, \quad \text{and} \quad c_1(M) < 0 \text{ outside a subvariety of } M\,.$$

Recall that the Kodaira dimension $K(M)$ is defined by

$$K(M) = \begin{cases} -\infty & \text{if} \quad N(M) = 0 \\ \max \dim \{\phi_{mk}\}(M) & \text{if} \quad N(M) \neq 0 \end{cases},$$

where $N(M) = \{m > 0 \mid H^0(M, K^m) = 0\}$ and ϕ_{mk} is the pluricanonical mapping. It is easy to see that $K(M) \leqslant$ the algebraic dimension of $M \leqslant n$. If $K(M) = n$, then M is called a manifold of general type.

In dimension two, surfaces can be classified bimeromorphically by their Kodaira dimension. The surfaces with $K(M) = -\infty$, 0 or 1 are well understood; moreover, $K(M) = 2$ (i.e., M is a surface of general type) if and only if M satisfies (**). Suppose M is a three-fold of general type and K is the canonical line bundle divisor. Kawatama [Ka] proved that if $K \cdot C \leqslant 0$ for every algebraic curve $C \subseteq M$, then M satisfies (**).

Most likely (**) always implies (*); that is, if M^n is an algebraic manifold with almost ample canonical line bundle, then the inequality (*) holds. This is not known for $n \geqslant 3$. One would also like to know what the relationship is between manifolds of general type and the inequality (**). In this respect, consider the following theorem of Siu [S5]. First recall that Siegel's theorem [Sg] says that for a complex manifold M^n, the transcendence degree of the meromorphic function field of M over $\mathbf{C}$ is less than or equal to n. When equality holds, M is called a Moishezon manifold. A Moishezon manifold can always be obtained by blowing up and down an algebraic manifold a finite number of times and hence is birational to some projective algebraic manifold. For a Moishezon manifold, there always exists a holomorphic vector bundle L over M such that

$c_1(L) \geqslant 0$ on M and $c_1(L) > 0$ outside some subvariety of M. Siu [S5] proved that the converse is also true under the weaker assumption that $c_1(L)$ is nonnegative everywhere and positive at some point. Thus, a manifold which satisfies (**) is Moishezon. It is also not known whether $\mathbf{CP}^n$, $n \geqslant 4$, can admit a nonstandard structure which is Moishezon. For $n = 3$, T. Peternell [Pe] proved that if M is a Moishezon 3-fold which is topologically isomorphic to $\mathbf{CP}^3$, then M is the standard $\mathbf{CP}^3$. His proof depends heavily on Mori's theory of extremal rays in 3-folds. One might expect that it is helpful for this problem to study rational curves in a Moishezon manifold which is a topological $\mathbf{CP}^n$.

5. Kähler-Einstein Metrics on Noncompact Manifolds

We now consider Kähler-Einstein metrics on complete noncompact manifolds. Let g be a complete Kähler-Einstein metric on M^n, i.e., $R_{i\bar{j}} = cg_{i\bar{j}}$ for some constant c. If $c > 0$, Myer's theorem would imply N is compact. Hence, $c \leqslant 0$ and $c_1(M) \leqslant 0$. In this section we consider the case $c_1(M) < 0$ and leave the case $c_1(M) = 0$ for the next section.

One would like to characterize noncompact manifolds which admit complete Kähler-Einstein metrics $g_{i\bar{j}}$ with $R_{i\bar{j}} = - g_{i\bar{j}}$. In particular, one would like to impose conditions on M to guarantee the existence and uniqueness of a Kähler-Einstein metric. First of all, uniqueness always holds. That is to say, if M and N are complete Kähler-Einstein manifolds with $R = -1$ and $F: M \to N$ is a biholomorphism, then F is an isometry. To prove this, let g and dv and g' and dv' denote the Kähler-Einstein metrics and volume forms of M and N, respectively. If we let $\rho = \log(F^*dv'/dv)$, then $\partial\bar{\partial}\rho = - f^*\,\mathrm{Ric}' + \mathrm{Ric} = F^*g' + g$. Taking traces, we have $\Delta\rho = -n + n \cdot e^{\rho/n}$. Hence, the maximum principle implies $\rho \leqslant 0$ and $F^*dv' \leqslant dv$. Replacing F by F^{-1}, we have $F^*dv' \geqslant dv$ and F is an isometry.

Uniqueness also holds for "almost" complete Kähler-Einstein metrics with scalar curvature equal to minus one. Here, a metric ds^2 on M is said to be almost complete if we can write M as an increasing union of domains Ω_α and there exist complete metrics ds_α^2 on Ω_α for each α such that ds_α^2 converges to ds^2 on compact subsets of M. See Cheng-Yau [C-Y1] for details.

We now consider the existence of Kähler-Einstein metrics with negative scalar curvature. Of course, the existence of such a metric would give restrictions on the complex structure of M. For example, Eiseman [Ei] proved that if there exists a Hermitian metric with scalar curvature less than a negative constant on M, then the pseudomeasure in the sense of Eiseman is in fact a measure, that is to say, M is measure hyperbolic.

In [C-Y1], Cheng and Yau obtained the existence of Kähler-Einstein metrics on a large class of noncompact manifolds. More precisely, they proved the following. Let M^n be a Hermitian manifold whose Ricci tensor defines a Kähler metric whose curvature and its covariant derivatives are bounded. Then M admits a Kähler-Einstein metric which is uniformly equivalent to the above metric.

If M admits a Hermitian metric with strongly negative Ricci curvature and is the increasing union of relatively compact, smooth, pseudoconvex open submanifolds, then there exists a unique (up to a scalar) almost complete Kähler-Einstein metric on M. Moreover, this metric is complete if M is complete.

In particular, there exists a complete Kähler-Einstein metric on any bounded domain in $\mathbf{C}^n$ which is the intersection of domains with C^2-boundaries. In the above statement, $\mathbf{C}^n$ can also be replaced by a Hermitian manifold with Ricci curvature bounded from above by a negative constant.

Mok and Yau [Mk-Y] proved that there exists a complete Kähler-Einstein metric on any bounded pseudoconvex domain in $\mathbf{C}^n$. This is the only known "canonical" metric on arbitrary bounded domains of holomorphy which is complete.

We now consider the case where the volume of M is finite. In this case, the "infinity" of M is very small (whereas the infinity of a bounded domain in $\mathbf{C}^n$ is quite large). The following is then conjectured: If the Ricci curvature is negative and M has finite topological type, then M can be compactified, that is, $M = \bar{M}/$(subvariety) for some compact Kähler manifold $\bar{M}$. In some cases, $\bar{M}$ is actually algebraic and hence M is quasi-projective.

For a locally Hermitian symmetric space M of finite volume, Baily and Borel [B-B], Satake [St] and Mumford [Mu] obtained (different) compactifications more or less explicitly. For these manifolds, Kähler-Einstein metrics exist. Siu and Yau [S-Y3] proved that a complete manifold, with finite volume with its curvature bounded between two negative constants, is quasi-projective.

If the above conjecture is true, then in studying Kähler manifolds with finite volume (and bounded covariant derivatives of the curvature) one need only consider $\bar{M}\backslash(D_1 \cup \cdots \cup D_k)$ where $\bar{M}$ is a compact Kähler manifold and $D_1, \ldots, D_k$ are connected divisors. If we have suitable algebraic data on how D_i looks like and how D_i intersects D_j, then one hopes that one may be able to construct Kähler-Einstein metrics on M. In dimension two, this is well understood. For example, suppose $C \subseteq \bar{M}^2$ is an elliptic curve and $C \cdot C < 0$. If s is a section of the bundle $[C]$ and $C = \{s=0\}$ then $dv_{\bar{M}}/|s|^2(\log|s|^2)^3$ is a complete asymptotic Kähler-Einstein metric on $\bar{M}/C$ with C as the cusps of the metric.

Suppose that D is a divisor on a compact Kähler manifold M satisfying $c_1(K+[D]) \geqslant 0$ on $\bar{M}$, $c_1(K+[D]) > 0$ on $\bar{M}\backslash D$ and $(K+[D]) - \varepsilon[D]|_D > 0$ then $\bar{M}\backslash D$ admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold $M = \bar{M}\backslash D$, a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes $\tilde{c}_i(M, D)$. The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes $\tilde{c}_1$ and $\tilde{c}_2$:

$$(*') \qquad (-1)^n\tilde{c}_1^{n-2} \cdot \tilde{c}_2 \geqslant \frac{(-1)^n}{2(n+1)}\, \tilde{c}_1^n .$$

A particularly significant fact is that equality holds in (*) if the quasi-projective manifold $\bar{M}\backslash D$ is the quotient of the unit ball in $\mathbf{C}^n$.

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \geqslant dv_{\textbf{Kähler}-\text{Einstein}} \geqslant c_2 dv_{\text{Caratheodory}}$$

where c_1 and c_2 are two universal positive constants. We have the following question: If the Caratheodory metric of M is complete, does M admit a complete Kähler-Einstein metric?

6. Ricci Flat Metrics on Noncompact Manifolds

We now consider Ricci flat metrics on a complete, noncompact manifold M. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose g and g' are two Ricci flat Kähler metrics on M. If they satisfy $g_{i\bar{j}} - g'_{i\bar{j}} = \partial\bar{\partial}F$ with F bounded, then $g_{i\bar{j}} = g'_{i\bar{j}}$. Note that in the compact case, the above condition means that g and g' belong to the same Kähler class. It also may be possible to drop the condition that F is bounded since there do not exist too many Ricci flat metrics.

In any case, the uniqueness problem is far from solved. Even when $M = \mathbf{C}^n$, Calabi proposed the following open problem: If $u: \mathbf{C}^n \to R$ is a strictly plurisubharmonic function with $\det\left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = 1$, then if the Kähler metric $ds_u^2 = \sum \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j$ is complete does it have zero curvature? Note that ds_u^2 is not complete in general. For example, Fatou and Bieberbach (see the book of Bochner and Martin [B-M], p. 45) gave a biholomorphism $F: \mathbf{C}^2 \to \Omega$, where $\Omega \subseteq \mathbf{C}^2$ is open and $\mathbf{C}^2/\Omega$ contains an open set, such that the Jacobian of F is identically equal to one. For $u = |z^1|^2 + |z^2|^2$, $ds_{u\cdot F}^2 = F^* ds_u^2 = F^* ds_0^2$ is not complete.

There are a lot of biholomorphisms F in $\mathrm{Aut}(\mathbf{C}^2)$ with Jacobian equal to one; for example, let $F(z, w) = (z + f(w), w)$ for any entire function f. For the above u, $u \circ F$ is still strictly plurisubharmonic and $ds_{u\cdot F}$ is complete and Ricci flat. Thus, intuitively, the larger the group $\mathrm{Aut}(M)$, the more difficult the problem is.

We now consider the question of existence. Just as in the case of negative scalar curvature, the existence of a complete, Ricci flat, Kähler metric will impose restrictions on the complex structure of M. For example, by the Schwarz lemma [Y4], we know that there does not exist any nontrivial holomorphic maps from M to a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant. As a corollary, if there exists a nontrivial holomorphic map from $\bar{M}$ to an algebraic curve of genus greater than one, then $M \subseteq \bar{M}$ cannot admit any complete Kähler metric with nonnegative Ricci curvature.

We conjecture that if M^n admits a complete Ricci flat Kähler metric, then $M = \bar{M}\backslash(\text{divisor})$ where $\bar{M}$ is compact and Kähler. This would mean that the infinity of M cannot be too large. Now suppose $M^2 = \bar{M}\backslash(\text{divisor})$ and dv is a Ricci flat volume form on M. One would like to determine M; by going to the universal cover, we can assume M is simply connected. Locally, $dv = (\sqrt{-1})^2 k dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2$ for some positive real function k. Since $\mathrm{Ric}(dv) = 0$, we have $\partial\bar{\partial}(\log k) = 0$ and k can be written as $k = |h|^2$ for some locally defined holomorphic function h. By a monodromy argument, we obtain a holomorphic 2-form $\eta = h dz^1 \wedge dz^2$, with h nowhere zero and $\eta \wedge \bar{\eta} = dv$. Hence $\eta^{-1} = h^{-1} dz^1 \wedge dz^2$ can be considered as a global section of the anti-canonical bundle K^{-1}.

Intuitively, one might expect that h approaches ∞ near the infinity of M and η^{-1} can be extended to $\bar{M}$, that is, there exists a nontrivial section

$S \in H^0(\bar{M}, K^{-1})$. This would imply that either K is trivial on M or $H^0(\bar{M}, K^n) = 0$ for every $m > 0$ and hence the Kodaira dimension of $\bar{M}^2$ would either be $-\infty$ or 0. This is because if $t \in H^0(\bar{M}, K^n)$, then $t \cdot S^n$ is a holomorphic function on M and hence constant; since S is zero somewhere unless K is trivial, we have $t \cdot S^n = 0$, so that $t = 0$ unless K is trivial on M.

Since M is Kähler and simply connected, the minimal model of $\bar{M}$ is a Kähler surface with $K = 0$ or $-\infty$ and $b_1 = 0$. When $K = 0$, it is either a $K-3$ surface or Enriques' surface. When $K = -\infty$ it is either a rational surface or a ruled surface of genus zero, $\bar{M}^2$ is equal the minimal model blown up successively at a finite number of points, and $M = \bar{M} \backslash \{s=0\}$ for some $0 \neq s \in H^0(\bar{M}, K^{-1})$. Conversely, if $M = \bar{M} \backslash \{s=0\}$ with $s \in H^0(\bar{M}, K^{-1})$ and $\bar{M}$ is as above, then M should admit a Ricci flat, complete, Kähler metric. In higher dimensions, the situation is much more complicated.

In physics, the following question has been studied. Is a Ricci flat metric with a suitable locally asymptotic property actually unique? This is the case when the metric is asymptotically flat. One would also like to know what happens when the metric is locally asymptotic to a cone. Perhaps assuming that the metric is Kähler may make this problem easier.

The existence of Ricci flat metrics has many applications. For example, using Ricci flat metrics, Siu [S1] proved that any surface M^2 with $c_1(M) = 0$ and $H^1(M, \mathbf{R}) = 0$ must be Kähler. See also Todorov [To] for higher dimensions. One can also ask the following question: Let M^{2n} be a simply-connected, compact, complex manifold where $n \geqslant 2$. If there exists a non-degenerate 2-form $\omega \in H^{2,0}(M)$, is M then Kähler? Todorov claimed that M is Kähler under an additional assumption: dim $H^{2,0}(M) = 1$.

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