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RADICALS AND HILBERT NULLSTELLENSATZ FOR NOT NECESSARILY ALGEBRAICALLY CLOSED FIELDS

by Dan LAKSOV

§ 1. THE MAIN RESULT AND DEFINITIONS

We shall in the following fix a (commutative) field k and denote by $\bar{k}$ the algebraic closure of k. Moreover, we shall denote by K a subfield of $\bar{k}$ containing k. The polynomial ring $k[x_1, x_2, ..., x_r]$ in r variables over k we denote by R. Given an ideal I in R, we denote by $Z_K(I)$ the algebraic subset of the r-dimensional affine space $\mathbf{A}_K^r$ which consists of the common zeroes of the polynomials in I. That is

$$Z_K(I) = \{(a_1, a_2, ..., a_r) \mid a_i \in K \quad \text{for} \quad i = 1, 2, ..., r \\ \text{and} \quad f(a_1, a_2 ..., a_r) = 0 \quad \text{for all} \quad f \in I\} .$$

The HILBERT NULLSTELLENSATZ is usually stated as follows:

Given an ideal I in R and a polynomial f of R, then f vanishes at all points of $Z_{\bar{k}}(I)$ if and only if $f^n \in R$ for some positive integer n.

In symbols the Hilbert Nullstellensatz can be written in the form:

$$\sqrt{I} = \{f \in R \mid Z_{\bar{k}}(f) \supseteq Z_{\bar{k}}(I)\} .$$

Here $\sqrt{I}$ denotes the *radical* of I, defined as the intersection of all prime ideals containing I, or equivalently by

$$\sqrt{I} = \{f \in R \mid f^n \in R \quad \text{for some positive integer} \quad n\} .$$

As an immediate consequence of the Hilbert Nullstellensatz we obtain the following result which is often referred to as the WEAK HILBERT NULLSTELLENSATZ:

Given an ideal I in R, then I is not all of R if and only if $Z_{\bar{k}}(I)$ is non-empty.

The Hilbert Nullstellensatz is one of the fundamental algebraic tools in geometry because it leads to a dictionary between algebraic subsets of $\mathbf{A}_{\bar{k}}^{r}$ on the one hand and radical ideals in R on the other. In algebraic geometry over a not-necessarily algebraically closed field K the main objects of study are the algebraic subsets of the affine space $\mathbf{A}_K^r$. However, if K is not algebraically closed, there always exists ideals in R with no zeroes in $\mathbf{A}_K^r$. Hence the above correspondence between radical ideals and algebraic sets fails, even in the sence of the weak Nullstellensatz.

The purpose of this article is to prove a generalization of the Hilbert Nullstellensatz which makes it possible to set up a dictionary between the algebraic subsets of $\mathbf{A}_K^r$ on the one hand and certain ideals of R, that we shall call *K-radical*, on the other.

To state our main result it is convenient to introduce the following notation:

Let $y_1, y_2, \ldots$ be a countably infinite set of elements that are algebraically independent over k. We denote by $P_K(m)$ the set of homogeneous polynomials in $k[y_1, y_2, \ldots, y_m]$ whose zeroes in $\mathbf{A}_K^m$, if any, are of the form $(a_1, a_2, \ldots a_{m-1}, 0)$. That is,

$$P_K(m) = \{p \in k[y_1, y_2, \ldots, y_m] \mid p \quad \text{is homogeneous and} \quad Z_K(p) \subseteqq Z_K(y_m)\}$$

Let A be a k-algebra and I an ideal of A. We denote by $\sqrt[K]{I}$ the subset

$$\{a \in A \mid \text{for some positive integer } m \text{ there exists a polynomial } p \in P_K(m) \text{ and elements } a_1, a_2, \ldots, a_{m-1} \text{ of } A \text{ such that } p(a_1, a_2, \ldots, a_{m-1}, a) \in I\}.$$

Below we shall prove that $\sqrt[K]{I}$ is an ideal in A which we call the *K-radical of I*. We can now state the main result of this article, which we shall refer to as the HILBERT K-NULLSTELLENSATZ as follows:

Given an ideal I of R, then

$$\sqrt[K]{I} = \{f \in R \mid Z_K(f) \supseteqq Z_K(I)\}.$$

As an immediate consequence of the Hilbert K-Nullstellensatz we obtain the following result which we refer to as the WEAK HILBERT K-NULLSTELLENSATZ:

Given an ideal I of R, then $\sqrt[K]{I}$ is not all of T if and only if $Z_K(I)$ is non-empty.

We observe that the Hilbert Nullstellensatz and its weak form are the Hilbert $\bar{k}$-Nullstellensatz and the weak Hilbert $\bar{k}$-Nullstellensatz. Indeed, it is clear that we have

$$P_{\bar{k}}(m) = \{1, y_m, y_m^2, y_m^3, \ldots\} \quad \text{for} \quad m = 1, 2 \ldots$$

Hence it follows from the definition of $\sqrt[K]{I}$ that, *if* $K = \bar{k}$ *then* $\sqrt[K]{I} = \sqrt{I}$ is the usual radical of I.

A result in the direction of the Hilbert Nullstellensatz was given by D. W. Dubois [2] and J. J. Risler [7] when k is ordered and K the real closure of k. A similar weaker result, which is however valid for any field k, when $k = K$, was given by W. A. Adkins, P. Gianni and A. Tognoli [1]. We shall return to these results in § 4 and see how they relate to the results of this article. In that section we also discuss some open problems related to the previous work.

In the process of generalizing the Hilbert Nullstellensatz we introduce, for each pair of fields k and K with $k \subseteqq K$, the *K-radical* of an ideal in any k-algebra. The K-radical of ideals in R makes it possible to give a treatment of the Nullstellensatz over an arbitrary field which is analogous to the traditional presentation over algebraically closed fields. Most properties that hold for the usual radical of an ideal can be seen to hold for the K-radical and the K-radical merits some interest of its own. Below we shall however only give those properties needed in our presentation of the Nullstellensatz. These properties we have collected in § 2. For a more complete treatment see Laksov [4] and [5].

The results of Dubois and Risler strongly suggest that the K-radical of an ideal can be defined by much smaller sets of polynomials than the sets $P_K(m)$. Restricting the set of polynomials used to define the K-radical would be the first step towards generalizing Hilberts 17'th problem and would give extremely interesting information about the fields involved. We shall however show that even modest advances in this direction may be very difficult. To be more precise we introduce, for each natural number m a set

$$P_K^0(m) = \{p \in k[y_1, y_2, ..., y_m] \mid p \text{ is homogeneous and the only zero of } p \text{ in } \mathbf{A}_K^m \text{ is the origin}\}$$

and for any ideal I in R we define a subset I_T of R by

$$I_T = \{f_i \in R \mid \text{for some positive integer } m \geqslant i \text{ there exists a polynomial } p \in P_K^0(m) \text{ and elements } f_1, f_2, ..., f_m \text{ in } R \text{ such that } p(f_1, f_2, ..., f_m) \in I\}$$

Then $P_K^0(m) \subseteq P_K(m)$ and consequently $I_T \subseteq \sqrt[K]{I}$. Moreover we have that $I \subseteq I_T$. The definition of I_T is apparently more natural and symmetric than that of $\sqrt[K]{I}$.

An intriguing problem raised by Tognoli is:

For which pair of fields $k \subseteq K$ *do we have that* $I_T = \sqrt[K]{I}$ *for all ideals* I *of* R?

It was long conjectured that equality holds for all pairs of fields (at least when the characteristics of k is zero). We shall however, in section 5, give examples showing that one may have strict inequality $I_T \subset \sqrt[K]{I}$ for the two pairs $k = K = \mathbf{Z}/2\mathbf{Z}$ and $k = K = \mathbf{Q}$.

Before we proceed (in § 3) to prove the Hilbert K-Nullstellensatz we shall in § 2 collect all the results that we need about the K-radicals and the polynomials $P_K(m)$ in the next section.

§ 2. Some properties of the K-radical

We shall denote by $S(m)$ the polynomial ring $k[y_1, y_2, \ldots, y_m]$.

Lemma 1. *Let* $p \in P_K(m)$ *and* $q \in P_k(n)$. *For each polynomial* $s = s(y_1, y_2, \ldots, y_{m+n}) \in S(m+n)$ *of degree one less than* q, *we have that,*

$$r = p(y_1 \cdot s, y_2 \cdot s, \ldots, y_{m-1} \cdot s, q(y_{m+1}, y_{m+2}, \ldots, y_{m+n})) \in P_K(m+n) .$$

Proof. It is clear that r is a homogeneous polynomial in $S(m+n)$. Let $(a_1, a_2, \ldots, a_{m+n}) \in \mathbf{A}_K^{m+n}$ be a zero of r. Since $p \in P_K(m)$, we have that $q(a_{m+1}, a_{m+2}, \ldots, a_{m+n}) = 0$. However, we have that $q \in P_K(m)$ so that $a_{m+n} = 0$. Consequently $r \in P_K(m+n)$ as asserted.

Proposition 2. *Let* A *be a k-algebra and* I *an ideal of* A. *Then the K-radical* $\sqrt[K]{I}$ *of* I *is an ideal of* A *(possibly* A *itself) which contains the radical of* I.

Proof. Since $P_K(1) = \{1, y_1, y_1^2, \ldots\}$ it is clear that the set $\sqrt[K]{I}$ contains $\sqrt{I}$.

Let f and g be elements in $\sqrt[K]{I}$. Then by the definition of the K-radical there are positive integers m and n, polynomials $p \in P_K(m)$ and $q \in P_K(n)$ and elements $f_1, f_2, \ldots, f_{m-1}$ and $g_1, g_2, \ldots, g_{n-1}$ of A such that

$$p(f_1, f_2, \ldots, f_{m-1}) \in I \quad \text{and}$$
$$q(g_1, g_2, \ldots, g_{n-1}, g) \in I$$

Let h be an element of A and let d be the degree of p. Then we have that

$$p(hf_1, hf_2, \ldots, hf_{m-1}, hf) = h^d p(f_1, f_2, \ldots, f_{m-1}, f) \in I .$$