2. Fixed points for homeomorphisms of the sphere

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2. Fixed points for homeomorphisms of the sphere

The next lemma is the second important ingredient. It can be proved by Nielsen's theory of fixed points. We will give a direct proof.

LEMMA 2.1. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a period 2 point which is not a fixed point, then the set* $\operatorname{Fix}(h)$ *can be written as a disjoint union* $\operatorname{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having point index equal to 1.*

Proof. Call x the point of period 2. Remark that since h preserve the orientation it induces on $\pi_1(\mathbf{S}^2 \backslash \{x, h(x)\}) = \mathbf{Z}$ the map $x \mapsto -x$. Choose an essential annulus $A \subset \mathbf{S}^2 \backslash \{x, h(x)\}$ large enough so that when we compose $h: A \to \mathbf{S}^2 \backslash \{x, h(x)\}$ with a retraction of $\mathbf{S}^2 \backslash \{x, h(x)\}$ on A we obtain a map $\bar{h}: A \to A$ which has no fixed point on the boundary, has the same fixed point as h and is equal to h in a neighborhood of the set of fixed points $\operatorname{Fix}(h) = \operatorname{Fix}(\bar{h})$. We will call $\tilde{A} \to A$ the universal cover of A of course $\tilde{A} = [0, 1] \times \mathbf{R}$ and if we denote by T a generator of the group of deck transformation of $\tilde{A} \to A$, we can write under this identification $T(x) = x + 1$ where addition is to be taken in the $\mathbf{R}$ coordinate. The map $\bar{h}$ lifts to a proper map $\tilde{h}$ which verifies $\tilde{h}T = T^{-1}\tilde{h}$. It follows that $\tilde{h}$ can be extended to the compactification of $\tilde{A}$ by its two ends $\varepsilon_-, \varepsilon_+$ by a map which exchange these to ends. Since $\tilde{A} \cup \{\varepsilon_-, \varepsilon_+\}$ is homeomorphic to a disk $\tilde{h}$ has a non empty compact set $\tilde{F}_1$ of fixed points which does not intersect the boundary because $\tilde{h}$ exchange ε_- and ε_+ and $\bar{h}$ has no fixed point on the boundary of A. Remark that the index of $\tilde{F}_1$ is 1. Moreover, the map $\tilde{A} \to A$ is injective on $\tilde{F}_1$ because if $\tilde{h}(x) = x$ we have $\tilde{h}(x+n) = x - n \neq x + n$ if $n \neq 0$. Since $\tilde{A} \to A$ is a covering it is clear that this map is also injective in a neighborhood of $\tilde{F}_1$. It follows that the image F_1 of $\tilde{F}_1$ under $\tilde{A} \to A$ is a compact non empty set of fixed points of $\bar{h}$ which has index 1. If $x \in \tilde{F}_1$, we have $T\tilde{h}(x+n) = T(x-n) = x + 1 - n \neq x + n$ for all n because $1/2 \notin \mathbf{Z}$. It follows that F_2, the image under $\tilde{A} \to A$ of $\operatorname{Fix}(T\tilde{h})$—which is also a compact non empty set of fixed points of $\bar{h}$ with index 1—is disjoint from F_1. If $x \in A$ is a fixed point of $\bar{h}$, it lifts to a point $\tilde{x} \in \tilde{A}$ which verifies $\tilde{h}(\tilde{x}) = \tilde{x} + n$. If $n = 2k$ then $\tilde{h}(\tilde{x}+k) = \tilde{h}(\tilde{x}) - k = \tilde{x} + 2k - k = \tilde{x} + k$. If $n = 2k - 1$ then $T\tilde{h}(\tilde{x}+k) = T(\tilde{x}+2k-1-k) = \tilde{x} + k$. This shows clearly that $\operatorname{Fix}(\bar{h}) = F_1 \cup F_2$. Since $\bar{h}$ is equal to h in a neighborhood of $\operatorname{Fix}(\bar{h}) = \operatorname{Fix}(h)$, this ends the proof. □

If we combine the Main Lemma 1.1 and lemma 2.1, we obtain:

LEMMA 2.2. *Let* $h: \mathbf{S}^2 \to \mathbf{S}^2$ *be an orientation preserving homeomorphism. If* h *has a non wandering point which is not a fixed point, then the set* $\mathrm{Fix}(h)$ *can be written as a disjoint union* $\mathrm{Fix}(h) = F_1 \cup F_2$ *with* F_1 *and* F_2 *closed non empty and having fixed point index equal to 1.*

Since we can compactify an orientation preserving homeomorphism of $\mathbf{R}^2$ by an orientation preserving homeomorphism of $\mathbf{S}^2$ with one more fixed point at infinity, we obtain the next two corollaries.

COROLLARY 2.3. (Brouwer's Lemma on translation arcs). *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be a fixed point free orientation preserving homeomorphism. Then* h *has no periodic point, each point wanders under* h. *Moreover, if* α *is a translation arc, the union* $\bigcup_{n \in \mathbf{Z}} h^n(\alpha)$ *is homeomorphic to a line and it does not accumulate on itself.*

COROLLARY 2.4. *Let* $h: \mathbf{R}^2 \to \mathbf{R}^2$ *be an orientation preserving homeomorphism. If the non wandering set of* h *is not reduced to the set of fixed points then there is a compact non empty subset* $F \subset \mathrm{Fix}(h)$ *which has fixed point index equal to 1.*