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THE COMPLEX CROSS RATIO ON THE HEISENBERG GROUP [1])

by A. Korányi and H. M. Reimann

Introduction

It is well known that the Heisenberg group H_n has a natural left-invariant distance function homogeneous under dilation. Under generalized stereographic projection H_n is then conformally equivalent with the sphere $S^{2n+1} \subset \mathbf{C}^{n+1}$ minus one point, equipped with the metric $d(z, w) = | 1 - z \cdot \bar{w} |^{1/2}$ (cf. [2], [3]).

It is known [3], [5] that the group of all conformal transformations of S^{2n+1} is a group $\tilde{G}$ whose identity component, to be denoted G, is isomorphic with $PSU(n+1, 1)$.

The number $1 - z \cdot \bar{w}$ can be regarded as a kind of "complex distance" on S^{2n+1}. There is a corresponding complex distance on H_n, whose absolute value equals the square of the natural distance function of H_n.

We shall call "complex cross ratio" the analogue of the classical cross ratio defined with the aid of the complex distance. Its absolute value is the cross ratio defined with the aid of the square of the (real) distance, we call it the "real cross ratio".

We will show that the largest group of transformations of S^{2n+1} (or equivalently of the one-point compactification of H_n) which preserves the complex cross ratio is G. For the real cross ratio the corresponding group is $\tilde{G}$.

The proof that these groups leave the cross ratio invariant is an easy verification. The converse is the more difficult part of the result. It could be proved by first making the rather easy verification that a transformation preserving the cross ratio is conformal in the sense of [3] and then using the highly non-elementary result that all conformal maps are in $\tilde{G}$. Instead of this we will give a simple direct proof.

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Besides giving this proof we will mention some interesting connections with the Szegö kernel function of the unit ball and we will prove some geometric results about the cross ratio.

Independently of the present investigation H. Furstenberg has defined, in group-theoretic terms, a cross ratio for general homogeneous spaces G/P where G is a semisimple Lie group and P a parabolic subgroup (private communication). It can be shown that in the case of $G = SU(n+1, 1)$ his cross ratio agrees with a power of our real cross ratio.

1. The Heisenberg group H_n is the set of pairs $[t, z] \in \mathbf{R} \times \mathbf{C}^n$ with the product

$$[t, z]\,[t', z'] = [t+t'+2\,\mathrm{Im}\, z \cdot \bar{z}', z+z']$$

where $z = (z_1, \cdots, z_n)$ and $z \cdot \bar{z}' = z_1\bar{z}'_1 + \cdots + z_n\bar{z}'_n$. The neutral element is $e = [0, 0]$. To every element $g = [t, z]$ we associate the complex number

$$A(g) = |z|^2 + it$$

and the *gauge* of g,

$$|g| = |A(g)|^{1/2} = (|z|^4 + t^2)^{1/4}.$$

The function $d(g', g) = |g^{-1}g'|$ defines a distance on H_n, left-invariant under translations. The complex number $A(g^{-1}g')$ can be regarded as a "complex distance" on H_n. We have, of course, $d(g', g) = |A(g^{-1}g')|^{1/2}$.

For a quadruple of distinct points g_1, g_2, g_3, g_4 in H_n we define the *complex cross ratio* by

$$(g_1, g_2, g_3, g_4) = \frac{A(g_3^{-1}g_1)}{A(g_4^{-1}g_1)} : \frac{A(g_3^{-1}g_2)}{A(g_4^{-1}g_2)}. \tag{1}$$

By continuity it can trivially be extended to the one-point compactification $H_n \cup \{\infty\}$. We will call its absolute value the *real cross ratio*; we have

$$|(g_1, g_2, g_3, g_4)| = \frac{d(g_1, g_3)^2}{d(g_1, g_4)^2} : \frac{d(g_2, g_3)^2}{d(g_2, g_4)^2}. \tag{2}$$

As in [2], for all $s > 0$ we denote by a_s the automorphism $[t, z] \to [s^2t, sz]$ of H_n; the group of all maps a_s is denoted by A. The group M of automorphisms of H_n is given by all unitary matrices m acting by $[t, z] \to [t, m \cdot z]$. The *inversion* h of H_n is defined on $H_n \setminus \{e\}$ by

$$h([t, z]) = \left[\frac{-t}{|z|^4 + t^2}, \frac{-z}{|z|^2 - it}\right].$$

The map $\tilde{m}$ is defined by

$$\tilde{m}([t, z]) = [-t, \bar{z}] .$$

All these actions extend trivially to $H_n \cup \{\infty\}$.

H_n acting on itself by left translations, together with A, M, and h generate a group G of transformations of $H_n \cup \{\infty\}$. The group generated by G and $\tilde{m}$ will be denoted $\tilde{G}$. It is not hard to see (cf. [2]) that G is isomorphic with $PSU(n+1, 1)$ and that $\tilde{G}$ has two connected components one of which is G.

Writing $g = [t, z]$, $g' = [t', z']$, we trivially have

(3) $$A(g^{-1}) = \overline{A(g)}$$

(4) $$A(gg') = A(g) + A(g') + 2z \cdot \bar{z}' .$$

Furthermore,

(5) $$A(a_s \cdot g) = s^2 A(g)$$

(6) $$A(m \cdot g) = A(g)$$

for all $a_s \in A$, $m \in M$, and

(7) $$A(\tilde{m} \cdot g) = \overline{A(g)} .$$

Writing g^*, g'^* instead of $h(g)$, $h(g')$, we have

(8) $$A(g^*) = \frac{1}{A(g)} .$$

Now (3) and (4) give

(9) $$A(g^{-1}g') = \overline{A(g)} + A(g') - 2z \cdot \bar{z}' .$$

Using this, the definition of g^*, and (8), we have

$$\begin{aligned} A(g^{*-1}g'^*) &= \overline{A(g^*)} + A(g'^*) - \frac{2z \cdot \bar{z}'}{\overline{A(g)}\, A(g')} \\ &= \frac{1}{\overline{A(g)}\, A(g')} \{A(g') + \overline{A(g)} - 2z \cdot \bar{z}'\} , \end{aligned}$$

and hence, using (3) again,

(10) $$A(g^{*-1}g'^*) = \frac{1}{A(g^{-1})A(g')} A(g^{-1}g') .$$

Proposition 1. *The complex cross ratio is invariant under* G *and the real cross ratio is invariant under* $\tilde{G}$.

Proof. It suffices to prove invariance under left translations by H_n (which is obvious) and then under the action of A, M, and h (and $\tilde{m}$ for the real case). From (5), (6), (10), and (7) invariance under each of these actions follows immediately.

Remark 1. Restricting to elements of the form $g = [0, z]$, $g' = [0, z']$, $|z| = |z'| = 1$, we have, immediately from (9),

$$A(g^{-1}g') = 2(1 - z \cdot \bar{z}') .$$

This is a "complex distance" on the sphere S^{2n-1}; it makes sense, of course, for every n, independently of any imbedding into an H_n. Generalized stereographic projection [2], [3]

$$\tag{11} \begin{aligned} \eta_0 &= \frac{i(1-|z|^2) - t}{i(1+|z|^2) + t} \\ \eta_j &= \frac{2z_j}{i(1+|z|^2) + t} \qquad (1 \leqslant j \leqslant n) \end{aligned}$$

maps $H_n \cup \{\infty\}$ onto the sphere

$$\sum_{j=0}^{n} |\eta_j|^2 = 1$$

in $\mathbf{C}^{n+1}$. A computation shows (cf. also the next Remark) that the complex cross ratio of H_n is then transferred to

$$\tag{12} (\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta^{(4)}) = \frac{1 - \eta^{(3)} \cdot \overline{\eta^{(1)}}}{1 - \eta^{(4)} \cdot \overline{\eta^{(1)}}} : \frac{1 - \eta^{(3)} \cdot \overline{\eta^{(2)}}}{1 - \eta^{(4)} \cdot \overline{\eta^{(2)}}} .$$

The theory of this cross ratio is therefore completely equivalent to the theory of our cross ratio on H_n. On this expression it is quite obvious (e.g. by using homogeneous coordinates) that it is invariant under the natural action of $SU(n+1, 1)$ on S^{2n+1}; this makes it possible to give an alternative proof of our Proposition through stereographic projection.

Remark 2. The equivalence of (1) and (12) as well as their G-invariance can also be proved without any explicit computation from some very general facts. As in [1], let $\mathscr{D}$ be a bounded symmetric domain and $D = c \cdot \mathscr{D}$ its image under the Cayley transformation c. Let $\mathscr{B}$ and B be the Shilov boundaries of $\mathscr{D}$ and D, respectively. Let $\mathscr{S}$ and S be the corresponding Szegö kernels. By continuity they extend to the boundary; more exactly, they can be regarded as functions on $\mathscr{B} \times \mathscr{B}$ and $B \times B$, undefined on some lower dimensional singular set. B can be identified with a nilpotent group N of affine linear automorphisms of $\mathscr{D}$. The transformation formula [1, (3.4)] of S under the full group of holomorphic automorphisms of D shows at once that the "cross ratio" defined on N by

$$\frac{S(g_3 \cdot 0, g_1 \cdot 0)}{S(g_3 \cdot 0, g_2 \cdot 0)} : \frac{S(g_4 \cdot 0, g_1 \cdot 0)}{S(g_4 \cdot 0, g_2 \cdot 0)} \tag{13}$$

is invariant. Furthermore, the relation

$$\mathscr{S}(z, w) = \frac{S(c \cdot 0, c \cdot 0)\, S(c \cdot z, c \cdot w)}{S(c \cdot z, c \cdot 0)\, S(c \cdot 0, c \cdot w)}$$

(see [1, p. 342]) immediately shows that under the inverse Cayley transform (13) is mapped into the cross ratio on $\mathscr{B}$ formed with the aid of $\mathscr{S}$. In the case where $\mathscr{D}$ is the unit ball in $\mathbf{C}^{n+1}$, N coincides with the Heisenberg group, and S, $\mathscr{S}$ are powers of the complex distance.

Before proving the converse of Proposition 1, we need the following lemmas.

LEMMA 1. *Let* $g = [t, z]$, $g' = [t', z'] \neq e$. *Then* $|gg'| = |g| + |g'|$ *if and only if* $t = t' = 0$ *and* $z = kz'$ $(k \geqslant 0)$.

Proof. By (4),

$$\begin{aligned} |gg'|^2 &= |A(gg')| = |A(g) + A(g') + 2z \cdot \bar{z}'| \\ &\leqslant |A(g)| + |A(g')| + 2|z|\,|z'| \\ &\leqslant |g|^2 + |g'|^2 + 2|g|\,|g'|. \end{aligned}$$

This last inequality is an equality if and only if $t = t' = 0$. In this case $A(g) = |z|^2$, $A(g') = |z'|^2$, and so the before last inequality is an equality if and only if $z = kz'$.

Definition. A curve $\sigma: \mathbf{R} \to H_n$ is called a *line* if $d(\sigma(s), \sigma(s')) = |s - s'|$ for all $s, s' \in \mathbf{R}$.

Note that this is not the same definition as that of Pansu [5, §68], since a different distance function is used. But the lines turn out to be the same as in [5] and will be used in a similar way.

LEMMA 2. *The lines* σ *such that* $\sigma(0) = e$ *are exactly the curves* $\sigma(s) = [0, sz^0]$ *with* $|z^0| = 1$.

Proof. Let σ be a line with $\sigma(0) = e$ and let $0 < r \leqslant s$. Then

$$|\sigma(s)| = s = r + (s-r) = |\sigma(r)| + |\sigma(r)^{-1}\sigma(s)|.$$

By Lemma 1 it follows that $\sigma(r) = [0, z(r)]$, $\sigma(r)^{-1}\sigma(s) = [0, kz(r)]$, and hence $\sigma(s) = [0, (1+k)z(r)]$. Setting $r = 1$ and $z^0 = z(1)$, and computing $|\sigma(s)|$, it follows that $z(s) = sz^0$ for $s \geqslant 1$.

Inverting the roles of r and s we get $\sigma(s) = [0, sz^0]$ for all $s \geqslant 0$. Finally, for all $s > 0$ we have

$$|\sigma(-s)^{-1}\sigma(s)| = 2s.$$

But $|\sigma(-s)^{-1}| = |\sigma(-s)| = |-s| = s$ and $|\sigma(s)| = s$. So Lemma 1 gives $\sigma(-s)^{-1} = [0, kz^0]$, and taking the gauge we get $k = s$. Hence $\sigma(-s) = [0, -sz^0]$, finishing the proof.

THEOREM. *If* γ *is a map of* $H_n \cup \{\infty\}$ *into itself preserving the real cross ratio then* $\gamma \in \tilde{G}$. *If* γ *preserves the complex cross ratio then* $\gamma \in G$.

Proof. In view of Proposition 1 and of the fact that $\tilde{m}$ changes the complex cross ratio to its conjugate it suffices to prove the first statement.

We may assume that $\gamma \cdot \infty = \infty$; in fact, if this is not already the case, we may compose γ with a left translation carrying $\gamma \cdot \infty$ to e and then with h. Composing, if necessary, with another left translation we may also assume $\gamma \cdot e = e$.

Now, for any $g, g' \neq e$, we have

$$\frac{|\gamma \cdot g|}{|\gamma \cdot g'|} = |(\gamma \cdot g, \gamma \cdot g', e, \infty)|^{1/2} = |(g, g', e, \infty)|^{1/2} = \frac{|g|}{|g'|}$$

and hence $|\gamma \cdot g| = \lambda |g|$ for all g with some $\lambda > 0$ independent of g. Composing γ with an appropriate element of A we may assume that $|\gamma \cdot g| = |g|$ for all g in H_n.

We also have

$$\frac{d(\gamma \cdot g, \gamma \cdot g')}{|\gamma \cdot g'|} = |(e, g, \infty, g')|^{1/2} = |(e, \gamma \cdot g, \infty, \gamma \cdot g')|^{1/2} = \frac{d(g, g')}{|\gamma \cdot g'|}$$

and hence $d(\gamma \cdot g, \gamma \cdot g') = d(g, g')$, i.e. γ is an isometry of H_n in the d-metric.

It follows that γ is surjective. In fact, $\gamma(H_n)$ is open by the invariance of domain theorem, and it is also closed since $p_k \in \gamma(H_n)$, $p_k \to p \in H_n$

implies $p_k = \gamma \cdot q_k$ with $|q_k| = |p_k|$ bounded, so there is a convergent subsequence $q_{k'} \to q$ and $p = \gamma \cdot q \in \gamma(H_n)$.

We denote by $\mathscr{L}_g$ the union of all lines through g. By Lemma 2, $\mathscr{L}_e$ is the z-hyperplane. Clearly $\mathscr{L}_g$ is gotten from $\mathscr{L}_e$ by left translation by g. It is then clear that the sets $\mathscr{L}_{[t,0]}$, $(t \in \mathbf{R})$ form a partition of H_n. Since isometries preserve lines, $\mathscr{L}_{\gamma \cdot [t,0]}$, $(t \in \mathbf{R})$ is also a partition of H_n. We claim that this implies $\gamma \cdot [t, 0] = [t', 0]$ with some t'. In fact, if $\gamma \cdot [t, 0] = [t', z']$ with $z' \neq 0$, then $\mathscr{L}_{\gamma \cdot [t,0]}$ contains $[t', z'] \cdot [0, z] = [0 + t' + 2 \operatorname{Im} z' \cdot \bar{z}, z' + z]$ for all $z \in \mathbf{C}^n$ and hence intersects $\mathscr{L}_{\gamma \cdot e} = \mathscr{L}_e$, which is a contradiction. Since γ is an isometry, we also have $t' = \pm t$ with the same sign for every $t \in \mathbf{R}$ by continuity. Composing, if necessary, with $\tilde{m}$, we may therefore assume that $\gamma \cdot [t, 0] = [t, 0]$ for all $t \in \mathbf{R}$. Together with what has already been said this also implies that γ maps each $\mathscr{L}_{[t,0]}$ onto itself.

Since γ preserves the z-hyperplane we may assume (composing with an element of M, which is transitive on the unit sphere of $\mathbf{C}^n$) that $\gamma \cdot [0, e_1] = [0, e_1]$, where $e_1 = (1, 0, \cdots, 0)$. Since γ preserves lines, $\gamma \cdot g(s) = g(s)$ for all $s \in \mathbf{R}$ with $g(s) = [0, se_1]$.

Let $g = [t, z]$ with $|z| = 1$. Since γ preserves $L_{[t,0]}$, we have $\gamma \cdot g = [t, z']$ with some z'. We have $d(g(s), \gamma \cdot g) = d(g(s), g)$, that is,

$$|1 + s^2 - 2sz_1 + it| = |1 + s^2 - 2sz'_1 + it'|$$

or, in other words,

$$\left| \frac{1+s^2}{2s} + i\frac{t}{2s} - z_1 \right| = \left| \frac{1+s^2}{2s} + i\frac{t}{2s} - z'_1 \right|$$

for all $0 \neq s, t \in \mathbf{R}$. Hence $z'_1 = z_1$.

This remark implies two things. First, γ fixes all elements of the form $[t_1, z_1 e_1]$ $(t \in \mathbf{R}, z_1 \in \mathbf{C})$. Second, γ maps the subset $T_1 = \{[0, z] \mid z_1 = 0\}$ into itself. Writing $e_2 = (0, 1, 0, \cdots, 0)$ we have that $\gamma \cdot [0, e_2]$ is in T_1, so there exists $m_2 \in M$ such that $m_2 \cdot [0, e_1] = [0, e_1]$ and $m_2 \gamma \cdot [0, e_2] = [0, e_2]$. Composing γ with this m_2 we can repeat the last argument and find that γ fixes all elements of the form $[t, z_1 e_1 + z_2 e_2]$ and maps the subset $T_2 = \{[0, z] \mid z_1 = z_2 = 0\}$ into itself. Continuing this process we can keep composing γ with elements of M until it becomes the identity map. This finishes the proof.

Remark. If one wants to prove only that the group preserving the complex cross ratio is G, this argument can be considerably simplified.

2. In this final section we show briefly how certain natural objects of the intrinsic geometry of S^{2n+1} (or equivalently of $H_n \cup \{\infty\}$) can be characterized with the aid of the complex cross ratio.

Mostow [4] calls **C**-spheres the intersections of S^{2n+1} with complex affine lines in $\mathbf{C}^{n+1}$; in order to distinguish them clearly from the **C**-hyperspheres (to be introduced later), we prefer to call them **C**-*circles*. They can be described as the sets $\{\alpha + s\beta \mid s \in \mathbf{C}, |s| = 1\}$ with fixed $\alpha, \beta \in \mathbf{C}^{n+1}$ such that $|\alpha|^2 + |\beta|^2 = 1$ and $\alpha \cdot \bar{\beta} = 0$. If $\alpha + s_j\beta\,(1 \leqslant j \leqslant 4)$ are four points on the same **C**-circle, then (12) and a little computation show that their cross ratio coincides with the classical cross ratio of the numbers s_j (hence, in particular, it is real). It is known [4] and easy to verify that G acts transitively on the set of all **C**-circles of non-zero radius.

In the classical case of **C** the cross ratio of four points is real if and only if the points lie on one circle. In our case we have the following result.

PROPOSITION 2. *Suppose* g_1, g_2, g_3, g_4 *are distinct points in* $H_n \cup \{\infty\}$. *Then*

(a) $(g_1, g_2, g_3, g_4) > 0$ *if and only if* g_3 *and* g_4 *are on the same orbit of the stabilizer of* g_1 *and* g_2.

(b) $(g_1, g_2, g_3, g_4) < 0$ *if and only if all the points lie on the same* **C**-*circle and* g_1, g_2 *separate* g_3 *from* g_4.

Proof. By G-invariance we may assume $g_1 = e$, $g_2 = \infty$. It is easy to see (cf. [2]) that the stabilizer of ∞ is MAH_n, hence the stabilizer of e and ∞ is MA.

Writing for a moment $g_j = [t_j, z_j], (j = 3, 4)$, the cross ratio now is

$$(e, \infty, g_3, g_4) = \frac{|z_3|^2 - it_3}{|z_4|^2 - it_4}$$

and this is real if and only if

$$|z_3|^2 t_4 - |z_4|^2 t_3 = 0$$

The expression is positive in the following situations:

$$\frac{|z_3|^2}{t_3} = \frac{|z_4|^2}{t_4} \qquad \text{with} \qquad t_3 \cdot t_4 > 0\,,$$

$$t_3 = t_4 = 0\,.$$

This is clearly the same as g_3 and g_4 being on the same MA-orbit.

Our expression is negative if and only if $z_3 = z_4 = 0$ and one of t_3 and t_4 is positive, the other negative. The t-axis completed by ∞ is a $\mathbf{C}$-circle, since it corresponds under (11) to the subset $\{(s, 0, \ldots, 0) \mid |s| = 1\}$ of S^{2n+1}. Since G is transitive on the set of $\mathbf{C}$-circles, the Proposition follows.

Remark. The generic MA-orbits are of codimension one in H_n. There are also singular orbits, corresponding to the case $z_3 = 0$ in our proof. Using the G-action it follows that these are exactly the connected proper subarcs of the $\mathbf{C}$-circles.

We shall call $\mathbf{C}$-*hyperspheres* the intersections of S^{2n+1} with complex affine hyperplanes of $\mathbf{C}^{n+1}$. It follows from the analogous statement about $\mathbf{C}$-circles (or directly) that G acts transitively on the set of all $\mathbf{C}$-hyperspheres of non-zero radius.

The equation of every (non-zero) $\mathbf{C}$-hypersphere can be written in the form

$$(\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta) = 1 \tag{14}$$

with some fixed $\eta^{(1)}, \eta^{(2)}, \eta^{(3)} \in S^{2n+1}$. In fact, when

$$\eta^{(1)} = e_0 = (1, 0, \cdots, 0), \quad \eta^{(2)} = -e_0 \quad \text{and} \quad \eta^{(3)} = e_1,$$

the left hand side of (14) reduces to $\eta_0 = 0$, which is indeed the equation of a $\mathbf{C}$-hypersphere. By transitivity of G the statement follows.

More generally, an equation

$$(\eta^{(1)}, \eta^{(2)}, \eta^{(3)}, \eta) = c \tag{15}$$

with $c \in \mathbf{C}$ always describes either a $\mathbf{C}$-hypersphere or the empty set. This is clear since by (12) the equation (15) is simply an inhomogeneous linear equation in the variable $\eta \in \mathbf{C}^{n+1}$. Our subsequent Proposition 3 will give some information about whether (15) gives the empty set, a point, or a genuine $\mathbf{C}$-hypersphere.

In the classical cases of $\mathbf{C}$ or $\mathbf{R}^n$ the conformal group is transitive on the set of triples of distinct points. In our case this is not so. The orbits of G on point triples can, however, be characterized with the aid of the cross ratio.

For fixed $-\pi/2 \leqslant \theta \leqslant \pi/2$ we denote by H_θ the union of the halfplane $\{s \in \mathbf{C} \mid \operatorname{Re}(e^{i\theta}s) \geqslant 0\}$ and $\{\infty\}$.

PROPOSITION 3. (a) *For fixed distinct* g_1, g_2, g_3, *the range of the function* $g \mapsto (g_1, g_2, g_3, g)$ *is a set* H_θ *with some* $\theta = \theta(g_1, g_2, g_3)$.

(b) *Two triples of distinct points* $\{g_1, g_2, g_3\}$ *and* $\{g'_1, g'_2, g'_3\}$ *can be transformed into each other by an element of* G *if and only if* $\theta(g_1, g_2, g_3) = \theta(g'_1, g'_2, g'_3)$.

Proof. Given g_1, g_2, g_3, we can as before transform g_1 and g_2 to e and ∞. The stabilizer of e and ∞ is MA; we can use an element of M to transform g_3 to the form $[t, pe_1]$ with $p > 0$, and then an element of A to transform it to $g^\theta = [\sin\theta, (\cos\theta)^{1/2} e_1]$ with some $-\pi/2 \leqslant \theta \leqslant \pi/2$. It is clear that two different elements of this form cannot be mapped onto each other by MA.

Writing $g = [t, z]$ we have

$$(e, \infty, g^\theta, g) = \frac{\cos\theta - i\sin\theta}{|z|^2 - it}.$$

The range of this, as g varies, is clearly H_θ. From this the Proposition follows immediately.

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