5. Remarks and examples

Let C and C_1 be two simple closed curves on F representing respectively the classes **C** and $\mathbf{C}_1$, in such a way that C and C_1 are in a position of minimum-intersection number.

Consider a neighborhood of the union of C and C_1 obtained by taking the union of a thin tubular neighborhood of each of these curves, and let C_2 denote the collection of those boundary curves of this neighborhood which are not null-homotopic.

Suppose first of all that C_2 is not empty. Then we have $s_1(C_2) = C_2$ and $s_2(C_2) = C_2$. (To see this, one can represent s_1 (respectively s_2) by an isometry of some hyperbolic metric, and then consider the geodesics g and g_1 in the classes of C and C_1. The isometry preserves the geodesics union $g \cup g_1$ and therefore it preserves an imbedded ε-neighborhood of that subset, and the boundary of the neighborhood). In this case, s_1 and s_2 have a common fixed point in **PMF**.

Suppose now that C_2 is empty. We have $s_1 \circ s_2(\mathbf{C}) = \mathbf{C}$ and $s_1 \circ s_2(\mathbf{C}_1) = \mathbf{C}_1$, and **C** and $\mathbf{C}_1$ have the property that for any element **F** in **MF**, we have either $i(\mathbf{F}, \mathbf{C}) \neq 0$ or $i(\mathbf{F}, \mathbf{C}_1) \neq 0$.

By assumption, $s_1 \circ s_2$ is reducible. Let n be an integer s.t. the map $(s_1 \circ s_2)^n$ preserves each component of the surface F cut along the reducing curve.

The mapping class $(s_1 \circ s_2)^n$ cannot have any pseudo-Anosov component, since if it had one, and if $\mathbf{F}^u$ denotes the class of the unstable foliation of that component, we have either $i(\mathbf{F}^u, \mathbf{C}) \neq 0$ or $i(\mathbf{F}^u, \mathbf{C}_1) \neq 0$. By the dynamics of a pseudo-Anosov (component) map on measured foliations space, the two classes of curves cannot be fixed by $s_1 \circ s_2$. Therefore, $s_1 \circ s_2$ cannot have pseudo-Anosov components.

So $(s_1 \circ s_2)^n$ has only finite order components.

By the same argument, $(s_1 \circ s_2)^n$ cannot have a non-trivial Dehn twist along a component of its reducing curve.

Therefore, $s_1 \circ s_2$ has only periodic components with no non-trivial Dehn twists along the reducing curve, so it is globally periodic, i.e. of finite order, a contradiction.

We conclude that $s_1 \circ s_2$ is pseudo-Anosov. This proves theorem 2.

5. Remarks and examples

1. We can easily classify now the structure of the group generated by two involutions:

Given the two involutions s_1 and s_2 of $M(F)$, the subgroup G they generate is an order-2 extension of the cyclic subgroup generated by $s_1 \circ s_2$. The elements of G that are not in that subgroup are all conjugate to s_1 or s_2. If s_1 and s_2 have a common fixed point in $\mathbf{T}$, the subgroup that they generate is finite. Otherwise, it is isomorphic to the infinite dihedral group $Z_2 * Z_2$.

2. In closing, we wish to point out that all three cases of Theorem 2 do in fact occur in every genus: To see that $s_1 \circ s_2$ can be of finite order we can take s_1 to be an horizontal rotation as in figure 2 and s_2 to be a vertical rotation as in figure 3. Since these rotations commute, $s_1 \circ s_2$ is an involution. (This example obviously generalizes to genus greater than two.)

To see that $s_1 \circ s_2$ can be reducible of infinite order, we can take s_1 to be a vertical rotation as in figure 3 and let $s_2 = s_1 \circ t_1 \circ t_b^{-1}$. Now s_2 is an involution by equation (1):

$$(17) \qquad (s_2)^2 = s_1 \circ t_1 \circ t_b^{-1} \circ s_1 \circ t_1 \circ t_b^{-1} = t_b \circ t_1^{-1} \circ t_1 \circ t_b^{-1} = 1\,.$$

Moreover, $s_1 \circ s_2 = t_1 \circ t_b^{-1} - 1$ which is a reducible map of infinite order. (Again, this example obviously generalizes to higher genera.)

To see that $s_1 \circ s_2$ can be pseudo-Anosov we can make a similar construction. Let s_1 be an involution. Suppose that A is a family of disjoint nontrivial simple closed curves. Let $B = s_1(A)$. Now suppose that A and B fill up F. Let t_A be the product of the Dehn twists about the components of A and t_B be the corresponding product associated to B. Let $s_2 = s_1 \circ t_A \circ t_B^{-1}$. As in the reducible case just described, s_2 is an involution. Furthermore, $s_1 \circ s_2 = t_A \circ t_B^{-1}$, which is a pseudo-Anosov map by an algorithm of Long's [6] generalizing Thurston's algorithm described in [4]. An example of this construction of case (iii) of Theorem 2 is depicted in figure 11, where s_1 is again the vertical rotation. (Again, this example easily generalizes.)

Alternatively, one can give a nonconstructive argument as follows. Let s_1 be a vertical rotation as in figure 3. Since $s_1(a_1) = b$, we know that s_1

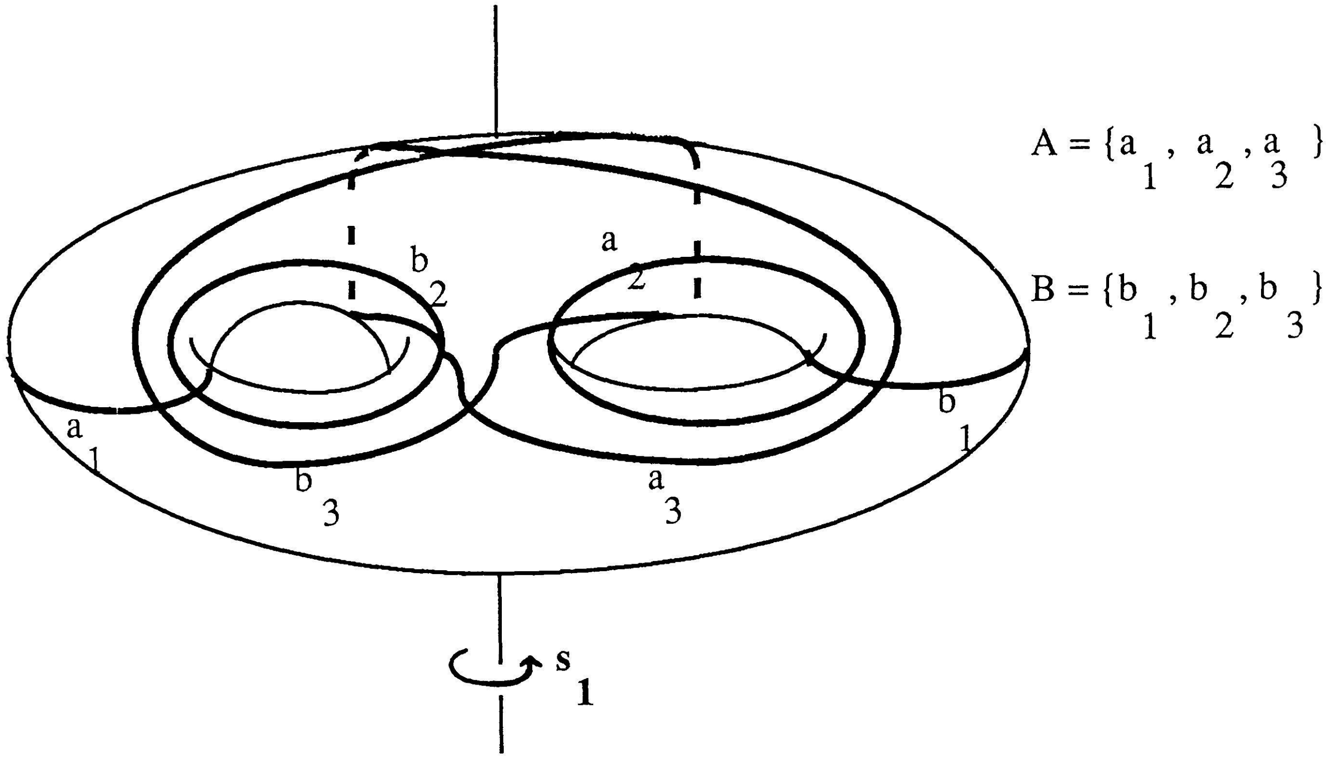

FIGURE 11.

is not in Fix(s_1). On the other hand, Fix(s_1) is clearly a closed set. Hence, we may find an open neighborhood of a_1 in $\mathbf{T} \cup \mathbf{PMF}$, U, such that U avoids Fix(s_1). Now, we may find a pseudo-Anosov, f, both of whose fixed points lie in U. (For example, this can be acheived by conjugating any given pseudo-Anosov by a sufficiently high power of t_1 .) Since Fix(s_1) is a compact set which avoids the repelling fixed point of f, it follows from the well known behavior of pseudo-Anosov maps on $\mathbf{T} \cup \mathbf{PMF}$ that $f^n(\mathrm{Fix}(s_1))$ is contained in U for sufficiently large n. Choose n subject to this condition and let $s_2 = f^n \circ s_1 \circ f^{-n}$. Finally, since Fix($s_2$) is equal to $f^n(\mathrm{Fix}(s_1))$, it follows from Theorem 2 that $s_1 \circ s_2$ is pseudo-Anosov.