4. Products of involutions

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **01.06.2024**

4. Products of involutions

Let s_1 and s_2 be two involutions. We are interested in the type of the element $s_1 \circ s_2$. This type will be seen to depend upon the intersection of the two sets $\mathrm{Fix}(s_1)$ and $\mathrm{Fix}(s_2)$, where $\mathrm{Fix}(s_i)$ denotes the fixed point set of s_i in the closed ball $\mathbf{T} \cup \mathbf{PMF}$.

THEOREM 2.

(i) $s_1 \circ s_2$ *is of finite order if and only if* $\mathrm{Fix}(s_1)$ *and* $\mathrm{Fix}(s_2)$ *have a common point in* $\mathbf{T}$.

(ii) *Suppose that* $s_1 \circ s_2$ *is not of finite order. If* $\mathrm{Fix}(s_1) \cap \mathrm{Fix}(s_2) \neq \emptyset$, *then* $s_1 \circ s_2$ *is reducible.*

(iii) $s_1 \circ s_2$ *is pseudo-Anosov if and only if* $\mathrm{Fix}(s_1)$ *and* $\mathrm{Fix}(s_2)$ *have empty intersection.*

Proof. (i) If s_1 and s_2 have a common fixed point in $\mathbf{T}$, then $s_1 \circ s_2$ also fixes this point and is therefore of finite order (cf. [4]).

For the converse, suppose that $s_1 \circ s_2$ is of finite order. Then by ([2], remarque p. 67), there is a point m in Teichmüller space such that m is fixed by $s_1 \circ s_2$.

The mapping classes s_1 and s_2 being involutions, we have $s_1(m) = s_2(m)$.

Now Teichmüller space has a metric, the Teichmüller metric (cf. [1]), for which the mapping class group acts by isometries. By Teichmüller's theorem, any two points in $\mathbf{T}$ can be joined by a unique geodesic. Each of the mapping classes s_1 and s_2 interchanges the points m and $s_1(m)$. Therefore, s_1 and s_2 fix the point which is at equal distance from m and $s_1(m)$, on the Teichmüller geodesic joining these points.

(ii) Let $\mathbf{F}$ be a common fixed point of s_1 and s_2 in $\mathbf{PMF}$. There exist two positive real numbers x_1 and x_2 such that if f is an element of $\mathbf{MF}$ in the class $\mathbf{F}$, then $s_1(f) = x_1 . f$ and $s_2(f) = x_2 . f$.

As s_1 and s_2 are of finite order, we have x_1 and $x_2 = 1$, so $s_1 \circ s_2(f) = f$. By ([2], exposé 9, §.III et IV), either $s_1 \circ s_2$ is of finite order or it is reducible.

(iii) Suppose that $\mathrm{Fix}(s_1) \cap \mathrm{Fix}(s_2)$ is empty. By (i), $s_1 \circ s_2$ is not of finite order. Suppose that it is reducible, and let $\mathbf{C}$ be the element of $\mathbf{MF}$ corresponding to the class of the reducing curve. We have $s_1(\mathbf{C}) = s_2(\mathbf{C})$. Let $\mathbf{C}_1$ denote the equivalence class $s_1(\mathbf{C})$.

Let C and C_1 be two simple closed curves on F representing respectively the classes $\mathbf{C}$ and $\mathbf{C}_1$, in such a way that C and C_1 are in a position of minimum-intersection number.

Consider a neighborhood of the union of C and C_1 obtained by taking the union of a thin tubular neighborhood of each of these curves, and let C_2 denote the collection of those boundary curves of this neighborhood which are not null-homotopic.

Suppose first of all that C_2 is not empty. Then we have $s_1(C_2) = C_2$ and $s_2(C_2) = C_2$. (To see this, one can represent s_1 (respectively s_2) by an isometry of some hyperbolic metric, and then consider the geodesics g and g_1 in the classes of C and C_1. The isometry preserves the geodesics union $g \cup g_1$ and therefore it preserves an imbedded ε-neighborhood of that subset, and the boundary of the neighborhood). In this case, s_1 and s_2 have a common fixed point in **PMF**.

Suppose now that C_2 is empty. We have $s_1 \circ s_2(\mathbf{C}) = \mathbf{C}$ and $s_1 \circ s_2(\mathbf{C}_1) = \mathbf{C}_1$, and $\mathbf{C}$ and $\mathbf{C}_1$ have the property that for any element $\mathbf{F}$ in **MF**, we have either $i(\mathbf{F}, \mathbf{C}) \neq 0$ or $i(\mathbf{F}, \mathbf{C}_1) \neq 0$.

By assumption, $s_1 \circ s_2$ is reducible. Let n be an integer s.t. the map $(s_1 \circ s_2)^n$ preserves each component of the surface F cut along the reducing curve.

The mapping class $(s_1 \circ s_2)^n$ cannot have any pseudo-Anosov component, since if it had one, and if $\mathbf{F}^u$ denotes the class of the unstable foliation of that component, we have either $i(\mathbf{F}^u, \mathbf{C}) \neq 0$ or $i(\mathbf{F}^u, \mathbf{C}_1) \neq 0$. By the dynamics of a pseudo-Anosov (component) map on measured foliations space, the two classes of curves cannot be fixed by $s_1 \circ s_2$. Therefore, $s_1 \circ s_2$ cannot have pseudo-Anosov components.

So $(s_1 \circ s_2)^n$ has only finite order components.

By the same argument, $(s_1 \circ s_2)^n$ cannot have a non-trivial Dehn twist along a component of its reducing curve.

Therefore, $s_1 \circ s_2$ has only periodic components with no non-trivial Dehn twists along the reducing curve, so it is globally periodic, i.e. of finite order, a contradiction.

We conclude that $s_1 \circ s_2$ is pseudo-Anosov. This proves theorem 2.

5. Remarks and examples

1. We can easily classify now the structure of the group generated by two involutions: