§12. The path from von Neumann algebras to knot polynomials

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Now, the computation of $D_A - d_B$ in the proof of theorem 10.1 shows that, if span $<L> = 4c$, one has $D_A - d_B = 4c$ and so $|A| + |B| = R$.

As Σ and Λ have no cut vertex, the proposition 11.12 implies that $A = S$ or $A = L$.

By lemma 11.9, this means that L is alternating. Q.E.D.

§ 12. The path from von Neumann algebras to knot polynomials

The discovery of the knot polynomials discussed here is due to Jones' investigations on von Neumann algebras, and not to the flourishing activity in low dimensional topology. In the light of previous work by J. Conway on Alexander's polynomial and of subsequent work by L. Kauffman (among others) on Jones' polynomial, such a genesis may seem unexpected. However this cannot be challenged, and should indeed appear rather as a delight of the subject than as any unpleasant awkwardness. With this point of view, we offer some guidelines for (some of) the surprising relationships put into light by V. Jones' work.

Factors of type II_1

An involution on a complex algebra M is a conjugate linear transformation $x \mapsto x^*$ of M such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in M$. The algebra $L(H)$ of all continuous operators on a Hilbert space H has a canonical involution, with x^* the adjoint of x, defined by $<x^*\xi | \eta> = <\xi | x\eta>$ for all $\xi, \eta \in H$. A representation of an involutive algebra M on H is a morphism of algebras $\pi: M \to L(H)$ with $\pi(x^*) = (\pi(x))^*$ for all $x \in M$. The algebra $L(H)$ carries several useful topologies, and in particular the weak topology, for which a sequence $(x_i)_{i \in I}$ of operators converges to 0 iff the numerical sequences $(<x_i\xi | \eta>)_{i \in I}$ converge to 0 for all pairs (ξ, η) of vectors in H.

A von Neumann algebra is an involutive algebra M with unit which has a faithful representation π on H with $\pi(1) = \text{id}$ and with $\pi(M)$ a weakly closed self-adjoint subalgebra of $L(H)$. (There are several equivalent definitions: see any textbook on the subject, for example one of [Di], [SZ], [Tak].) A von Neumann algebra is defined to be a *factor of type II_1* if

(1) The center of M is reduced to scalar multiples of 1.

(2) There exists a normalized finite trace, namely a linear form $\text{tr}: M \to \mathbf{C}$ with $\text{tr}(1) = 1$ and $\text{tr}(xy) = \text{tr}(yx)$ for all $x, y \in M$.

(3) The dimension of M over $\mathbf{C}$ is infinite.

Moreover, if M is a factor of type II_1:

(4) There exists a *unique* normalized finite trace.

(5) For any real number $d \in [0, 1]$, there exists a self-adjoint idempotent $e \in M$ with $\mathrm{tr}(e) = d$.

(6) The trace is positive and faithful: $\mathrm{tr}(x^*x) \geqslant 0$ for all $x \in M$, with equality for $x = 0$ only.

(7) The algebra M is simple. In particular, any representation of M is faithful.

Let us add three comments. The notion of trace used in (2) may seem slightly unusual in the context of operator algebras, but is the same as the standard notion because we consider factors of type II_1 only; see [FH]. Because of (5), factors of type II_1 are also called finite and continuous. Concerning (7), the following may be added under suitable separability assumptions: Murray and von Neumann have defined for any representation of M a multiplicity, which is a positive number (possibly infinite), and two representations of M are unitarily equivalent iff they have the same multiplicity.

A factor M of type II_1 is said to be *hyperfinite* if it has the following property: for any integer $n \geqslant 1$, for any sequence $x_1, ..., x_n \in M$ and for any $\varepsilon > 0$, there exists a finite dimensional self-adjoint subalgebra K of M such that

$$d_2(x_j, K) < \varepsilon, \qquad j = 1, ..., n$$

where d_2 is the distance associated to the norm $x \mapsto \mathrm{tr}(x^*x)^{1/2}$ on M. Murray and von Neumann showed that two hyperfinite factors of type II_1 which can be represented on a separable Hilbert space are *-isomorphic; the standard notation for "the" hyperfinite factor of type II_1 is R. Moreover, they showed that any factor of type II_1 contains a copy of R [MN]. Instead of "hyperfinite", the factor R is also called "approximately finite dimensional", "injective", "semi-discrete" or "amenable", and there is a good reason for each of these words. A sub-factor of R is either finite dimensional or isomorphic to R itself [Co_1]. The importance of R in the theory cannot be overemphasized.

Consider for example a countable group Γ, the Hilbert space $l^2(\Gamma)$ of complex functions $\xi : \Gamma \to \mathbf{C}$ with $\sum_{g \in \Gamma} | \xi(g) |^2 < \infty$, the right regular representation $\rho : \Gamma \to L(l^2(\Gamma))$ defined by $(\rho(g)\xi)(h) = \xi(hg)$, and the algebra

$W^*(\Gamma)$ of operators x on $l^2(\Gamma)$ such that $x\rho(g) = \rho(g)x$ for all $g \in \Gamma$. It can be shown that $W^*(\Gamma)$ is the von Neumann algebra generated by $\lambda(g)$ for $g \in \Gamma$, where $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$. If all conjugacy classes (other than $\{1\}$) in Γ are infinite, then $W^*(\Gamma)$ is a factor of type II_1; moreover it makes sense to write any element in $W^*(\Gamma)$ as a (usually infinite) sum $\sum_g z_g \lambda(g)$, and the normalised trace of such an element is z_1. Assuming that Γ has infinite conjugacy classes and moreover that Γ contains an element a of infinite order, we may formulate a nice exercise to illustrate property (5) above: for any $d \in [0, 1]$, show that the infinite sum

$$d + \sum_{\substack{n \in \mathbf{Z} \\ n \neq 0}} \frac{\sin(dn\pi)}{n\pi} \lambda(a^n)$$

defines in $W^*(\Gamma)$ a self-adjoint idempotent of normalized trace d (solution in [Au]).

If Γ has infinite conjugacy classes and is moreover amenable, then $W^*(\Gamma)$ is a model for the hyperfinite factor R, by [Co_1]. Examples of amenable groups: the group of permutations with finite supports of a countable set, or any solvable group.

To cut a long story short, Murray and von Neumann knew of two non isomorphic factors of type II_1, namely R and $W^*(\Gamma)$ for Γ the non abelian free group on two generators [MN]. J. Schwartz established the existence of a third one twenty years later [Sc], and D. McDuff showed there are uncountably many [McD]. During the 1970's, A. Connes made several break-throughs in the knowledge of factors; for a review of the subject before 1980, see [Co_2]. By then, it was reasonable for V. Jones to embark in the study of *relative* problems: understand subfactors (of type II_1) in a given factor of type II_1.

THE INDEX

Let $M_0 \subset M_1$ be a pair of factors of type II_1. It is natural to look for invariants of these data, with respect to conjugacy of M_0 by (possibly inner) automorphisms of M_1. For the present discussion, the most successful invariant is the *index* $[M_1 : M_0] \in [1, \infty]$. Its definition appears in [Jo_1] and [Jo_2]; see also below.

Once the index is defined, the most obvious problem is to compute exactly its possible values. If M_1 is the hyperfinite factor of type II_1, then the set of possible values $[M_1 : M_0]$ consists of a continuous spectrum $[4, \infty]$,

a discrete spectrum $\{4 \cos^2(\pi/n)\}_{n=3,4,5,\ldots}$.

This was quite a surprise at the time, as continuity is so often the rule for objects defined by M_1. (If the factor M_1 is not hyperfinite, our knowledge is fragmentary and the possible values for $[M_1 : M_0]$ may constitute a proper subset of the spectrum just described. See [PP].)

Let us now define the index and indicate some steps in the proof of Jones' result about its spectrum. Given a pair $M_0 \subset M_1$, there exists a *conditional expectation* $e_1 : M_1 \to M_0$ which is a projection such that $e_1(axb) = ae_1(x)b$ and $\mathrm{tr}(e_1(x)) = \mathrm{tr}(x)$ for $a, b \in M_0$ and $x \in M_1$. In fact both e_1 and elements in M_1 may be looked at as operators on the Hilbert space $L^2(M_1, \mathrm{tr})$ obtained by completion of M_1 for the scalar product $<x | y> = \mathrm{tr}(x^* y)$; then e_1 is the orthogonal projection of M_1 onto M_0, and $x \in M_1$ acts on $L^2(M_1, \mathrm{tr})$ as the extension of the multiplication $y \mapsto xy$.

Thus it makes sense to consider the von Neumann algebra M_2 generated by e_1 and M_1. With one exception which is precisely the case in which $[M_1 : M_0] = \infty$, the algebra M_2 is again a factor of type II_1. In the later case, the definition of the index is

$$[M_1 : M_0] = \frac{1}{\mathrm{tr}_2(e_1)}$$

where tr_2 denotes the trace on M_2.

As $M_1 \subset M_2$ is again a pair as above, the same construction may be iterated, and one obtains a *tower*

$$M_0 \subset M_1 \subset \ldots \subset M_n \subset M_{n+1} = <M_n, e_n> \subset \ldots$$

of factors of type II_1. A basic fact is that the e_i's satisfy three types of relations

$$\begin{array}{lll} \textit{idempotence:} & e_i^2 = e_i, & \\ \textit{braiding:} & e_i e_{i\pm1} e_i = [M_1 : M_0]^{-1} e_i, & \\ \textit{commutation:} & e_i e_j = e_j e_i & \text{if} \quad |i-j| \geqslant 2. \end{array}$$

Also the traces on the M_n's induce a trace tr on the algebra generated by the e_i's with

Markov property: $\mathrm{tr}(we_i) = [M_1 : M_0]^{-1} \mathrm{tr}(w)$ for w in the algebra generated by $M_0, e_1, \ldots e_{i-1}$.

The invocation of Markov here refers to the property of the trace: its value on each step $M_{n+1} = <M_n, e_n>$ is readily computable in terms of the trace on the previous step M_n. There is moreover the crucial tool of

positivity: the algebra of operators generated by the e_i's has an involution $w \mapsto w^*$ and $\mathrm{tr}(w^*w) > 0$ for any $w \neq 0$ in this algebra.

An analysis of these properties shows that, in case the index is smaller than 4, then only the discrete spectrum.

$$[M_1 : M_0] \in \{4\cos^2(\pi/n)\}_{n \geqslant 3}$$

is permitted. (The reader will have some flavour of the analysis if he solves the following exercise: consider four unit vectors $e_1, ..., e_4$ in the usual 3-space such that the scalar products satisfy

$$<e_1 | e_2> \;=\; <e_2 | e_3> \;=\; <e_3 | e_4> \;=\; \cos\varphi$$
$$<e_1 | e_3> \;=\; <e_1 | e_4> \;=\; <e_2 | e_4> \;=\; 0$$

for some angle φ; then $\cos\varphi = 1/2(\sqrt{5}-1)$ and φ can only be one of two possible angles.)

Constructing pairs with $[M_1 : M_0] \geqslant 4$ turns out to be easy (at least when M_1 is hyperfinite). For the discrete spectrum, consider first a complex number $\beta \neq 0$, an integer $n \geqslant 1$, and the algebra $\mathscr{A}_{\beta, n}$ abstractly defined (as a complex associative algebra) by

$$\text{generators:} \quad 1, \varepsilon_1, ..., \varepsilon_{n-1},$$

$$\text{relations:} \quad \begin{cases} \varepsilon_i^2 = \varepsilon_i, \\ \varepsilon_i \varepsilon_{i \pm 1} \varepsilon_i = \beta^{-1} \varepsilon_i, \\ \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i \quad \text{if} \quad |i-j| \geqslant 2. \end{cases}$$

If $\beta > 0$, the construction of a pair with $[M_1 : M_0] = \beta$ reduces to finding a representation of $\mathscr{A}_{\beta, \infty} = \lim\limits_{n \to \infty} \mathscr{A}_{\beta, n}$ by operators on a Hilbert space with each ε_i self-adjoint. Manipulations of linear algebra show that this can be done precisely when β is in the spectrum of indices; see Jones' papers, as well as the expository [GHJ].

Note finally that the e_i's and the ε_i's should not be confused: Given some pair $M_0 \subset M_1$ of index β, it is of course obvious that $\mathscr{A}_{\beta, n}$ maps onto the algebra generated by $1, e_1, ..., e_{n-1}$. But for β in the discrete spectrum, this map has a non trivial kernel when n is large enough.

HECKE ALGEBRAS AND POLYNOMIALS

One of the main points to retain from above is the following: an interesting problem with a surprising solution in the theory of von Neumann algebras has motivated a serious study of the algebras $\mathcal{A}_{\beta,n}$. Now $\mathcal{A}_{\beta,n}$ appears to be in close relationship with

(a) Artin's braid group B_n with generators $\sigma_1, ..., \sigma_{n-1}$ and relations as in § 6.

(b) The Hecke algebra of § 4, that we denote from now on by $H_{q,n}$ to stress the dependence on q, where parameters fit well if $\beta = 2 + q + q^{-1}$.

To make this relationship transparent, we turn to another presentation of $\mathcal{A}_{\beta,n}$. Choose a complex number q with $\beta = 2 + q + q^{-1}$ (observe that $q \neq -1$ as $\beta \neq 0$) and set

$$T_i = q\varepsilon_i - (1-\varepsilon_i) \quad \text{so that} \quad \varepsilon_i = \frac{T_i + 1}{q + 1}$$

for $i = 1, ..., n-1$. Then a straightforward computation shows that $\mathcal{A}_{\beta,n}$ has a presentation with generators $T_1, ..., T_{n-1}$ and relations

(1) $T_i^2 = (q-1)T_i + q$,

(2) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$,

(3) $T_i T_j = T_j T_i$ if $|i-j| \geqslant 2$,

(S) $T_i T_{i+1} T_i + T_i T_{i+1} + T_{i+1} T_i + T_i + T_{i+1} + 1 = 0$.

The last relation was first pointed out by R. Steinberg. One has now more precisely:

(a) The assignment $\sigma_i \mapsto T_i$ extends to a homomorphism ρ_q from B_n to the invertible elements of $\mathcal{A}_{\beta,n}$ (compare with § 6).

(b) $\mathcal{A}_{\beta,n}$ is the quotient of the Hecke algebra $H_{q,n}$ of § 4 by the relation (S).

For infinitely many values of q (namely $q \in \mathbf{R}$ and $q \geqslant 1$, corresponding to $\beta \geqslant 4$), Jones knew from his study of factors [Jo$_2$] that $\mathcal{A}_{\beta,n}$ is given with a faithful positive Markov trace tr. For each braid $\alpha \in B_n$, he set

$$V_\alpha(q) = -\left(\frac{q+1}{q^{1/2}}\right) q^{e/2} \operatorname{tr}(\rho_q(\alpha))$$

where e is the exponent sum of α as a word on the σ_i's. The first theorem in [Jo$_3$] is that V_α depends only on the link $K(\alpha)$ obtained by closing α. Also $V_\alpha(q)$ [respectively $q^{1/2}V_\alpha(q)$] is a Laurent polynomial in q

if $K(\alpha)$ has an odd [resp. even] number of components; in particular $V_\alpha(q)$ can be defined for any $q \in \mathbf{C}$, not just for those corresponding to good traces on some $\mathscr{A}_{\beta,n}$. And, most importantly for the early growth of the subject, a computation in the summer 1984 with the trefoil knot showed that V is not a mere variant of the Alexander polynomial. In fact, during a few hours, this was thought to reveal a mistake in computations! See end of § 7 for more details on the independence of the polynomials.

One way to recover the two variable polynomial is to introduce a *family* of traces on $H_{q,\infty} = \lim_{n\to\infty} H_{q,n}$, indexed by a complex parameter z. This programme was pursued by Ocneanu, and exposed in §§ 5-6 above. Observe that

(1) Only one of Ocneanu's traces pass to the quotient $\mathscr{A}_{\beta,\infty}$, namely that corresponding to $z = q(q+1)^{-2}$.

(2) Ocneanu's traces are positive for some values of the pair (q, z) only: the picture appears in Wenzl's thesis [We] and also in [Jo$_4$].

(3) It does help to keep positivity considerations in mind when studing knot polynomials: see § 14 in [Jo$_5$].

Added in proof

1. V. Turaev has another and simpler proof of some of the geometric arguments given in § 11. See a next issue of this journal.

2. K. Murasugi has informed us that he has now proved conjecture *C*.

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