

# §1. Systems of Differential Equations (See [O], [Bj])

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0.5. In the situation of § 0.2,  $u(x) = c_0(x)f(x)^s + \dots$  satisfies  $\text{supp sp}(u(x)) = \{\pm df(x)\}$ . Therefore  $P_m(x, df)$  must be zero if  $P(x, \partial)u(x) = 0$ . In fact otherwise the bijectivity of  $P: \mathcal{C}_M \rightarrow \mathcal{C}_M$  implies  $\text{sp}(u) = 0$ .

0.6. Such a method of studying functions or differential equations locally on the cotangent bundle is called microlocal analysis. After Sato's discovery of microfunctions, microlocal analysis was studied intensively in Sato-Kawai-Kashiwara [SKK].

Also L. Hörmander [H] worked in the  $C^\infty$ -case. Since then, microlocal analysis has been one of the most fundamental tools in the theory of linear partial differential equations.

## § 1. SYSTEMS OF DIFFERENTIAL EQUATIONS (See [O], [Bj])

1.1. Let  $X$  be a complex manifold. A system of linear differential equations can be written in the form

$$(1.1.1) \quad \sum_{j=1}^{N_0} P_{ij}(x, \partial)u_j = 0, \quad i = 1, 2, \dots, N_1.$$

Here  $u_1, \dots, u_{N_0}$  denote unknown functions and the  $P_{ij}(x, \partial)$  are differential operators on  $X$ . The holomorphic function solutions of (1.1.1) are simply the kernel of the homomorphism

$$(1.1.2) \quad P: \mathcal{O}_X^{N_0} \rightarrow \mathcal{O}_X^{N_1}$$

which assigns  $(v_1, \dots, v_{N_1})$  to  $(u_1, \dots, u_{N_0})$ , where  $v_i = \sum_j P_{ij}(x, \partial)u_j$ .

Let us denote by  $\mathcal{D}_X$  the ring of differential operators with holomorphic coefficients. Then

$$(1.1.3) \quad P: \mathcal{D}_X^{N_1} \rightarrow \mathcal{D}_X^{N_0}$$

given by  $(Q_1, \dots, Q_{N_1})$  to  $(\sum Q_i P_{i1}, \dots, \sum Q_i P_{iN_0})$  is a left  $\mathcal{D}_X$ -linear homomorphism. If we denote by  $\mathcal{M}$  the cokernel of (1.1.3), then  $\mathcal{M}$  becomes a left  $\mathcal{D}_X$ -module and  $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X)$  is the kernel of (1.1.2). This means that the set of holomorphic solutions to  $Pu = 0$  depends only on  $\mathcal{M}$ .

For this reason we mean by a system of linear differential equations a left  $\mathcal{D}_X$ -module.

1.2. Let us take a local coordinate system  $(x_1, \dots, x_n)$  of  $X$ . Then any differential operator  $P$  can be written in the form

$$(1.2.1) \quad P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x) \partial^\alpha$$

where  $\partial^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$  and the  $a_\alpha(x)$  are holomorphic functions. For  $j \in \mathbf{N}$ , we set

$$P_j(x, \xi) = \sum_{|\alpha|=j} a_\alpha(x) \xi^\alpha,$$

where  $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ , and we call  $\{P_j(x, \xi)\}$  the *total symbol* of  $P$ . The largest  $m$  such that  $P_m \neq 0$  is called the *order* of  $P$  and  $P_m$  is called the *principal symbol* of  $P$  and denoted by  $\sigma(P)$ .

Let us denote by  $T^*X$  the cotangent bundle of  $X$ , and let

$$(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$$

be the associated coordinates of  $T^*X$ . It is a classical result that if we consider  $\sigma(P)$  as a function on  $T^*X$ , then this does not depend on our choice of the local coordinate system  $(x_1, \dots, x_n)$ .

1.3. Let  $M$  be a real analytic manifold, and  $X$  its complexification, e.g.,  $M = \mathbf{R}^n \subset X = \mathbf{C}^n$ . Let  $P$  be a differential operator on  $X$ . When  $\sigma(P)(x, \xi) \neq 0$  for  $(x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus \{0\})$ ,  $P$  is called an elliptic differential operator. In this case, we have the following result.

**PROPOSITION 1.3.1.** *If  $u$  is a hyperfunction (or distribution) on  $M$  and  $Pu$  is real analytic, then  $u$  is real analytic. More precisely if we denote by  $\mathcal{A}$  the sheaf of real analytic functions on  $M$  and by  $\mathcal{B}$  (resp.  $\mathcal{D}b$ ) the sheaf of hyperfunctions (resp. distributions) on  $M$ , then  $P: \mathcal{B}/\mathcal{A} \rightarrow \mathcal{B}/\mathcal{A}$  (resp.  $P: \mathcal{D}b/\mathcal{A} \rightarrow \mathcal{D}b/\mathcal{A}$ ) is a sheaf isomorphism.*

This suggests that if  $\sigma(P)(x, \xi) \neq 0$ , we can consider the inverse  $P^{-1}$  in a certain sense. Since  $(x, \xi)$  is a point of the cotangent bundle,  $P^{-1}$  is attached to the cotangent bundle.

In fact, as we shall see in the sequel, we can construct a sheaf of rings  $\mathcal{E}_X$  on  $T^*X$  such that  $\mathcal{D}_X \subset \pi_* \mathcal{E}_X$ , where  $\pi$  is the canonical projection  $T^*X \rightarrow X$ . Moreover if  $P \in \mathcal{D}_X$  satisfies  $\sigma(P)(x, \xi) \neq 0$  at a point  $(x, \xi) \in T^*X$ , then  $P^{-1}$  exists as a section of  $\mathcal{E}_X$  on a neighborhood of  $(x, \xi)$ .

This can be compared to the analogous phenomena for polynomial rings, as shown in the following table.