§1. Systems of Differential Equations (See [O], [Bj])

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0.5. In the situation of § 0.2, $u(x) = c_0(x)f(x)^s + \ldots$ satisfies supp $\mathrm{sp}(u(x)) = \{\pm df(x)\}$. Therefore $P_m(x, df)$ must be zero if $P(x, \partial)u(x) = 0$. In fact otherwise the bijectivity of $P: \mathscr{C}_M \to \mathscr{C}_M$ implies $\mathrm{sp}(u) = 0$.

0.6. Such a method of studying functions or differential equations locally on the cotangent bundle is called microlocal analysis. After Sato's discovery of microfunctions, microlocal analysis was studied intensively in Sato-Kawai-Kashiwara [SKK].

Also L. Hörmander [H] worked in the C^∞-case. Since then, microlocal analysis has been one of the most fundamental tools in the theory of linear partial differential equations.

§ 1. Systems of Differential Equations (See [O], [Bj])

1.1. Let X be a complex manifold. A system of linear differential equations can be written in the form

$$\sum_{j=1}^{N_0} P_{ij}(x, \partial)u_j = 0\,, \quad i = 1, 2, \ldots, N_1\,. \tag{1.1.1}$$

Here $u_1, \ldots, u_{N_0}$ denote unknown functions and the $P_{ij}(x, \partial)$ are differential operators on X. The holomorphic function solutions of (1.1.1) are simply the kernel of the homomorphism

$$P: \mathscr{O}_X^{N_0} \to \mathscr{O}_X^{N_1} \tag{1.1.2}$$

which assigns $(v_1, \ldots, v_{N_1})$ to $(u_1, \ldots, u_{N_0})$, where $v_i = \sum_j P_{ij}(x, \partial)u_j$.

Let us denote by $\mathscr{D}_X$ the ring of differential operators with holomorphic coefficients. Then

$$P: \mathscr{D}_X^{N_1} \to \mathscr{D}_X^{N_0} \tag{1.1.3}$$

given by $(Q_1, \ldots, Q_{N_1})$ to $(\Sigma Q_i P_{i1}, \ldots, \Sigma Q_i P_{iN_0})$ is a left $\mathscr{D}_X$-linear homomorphism. If we denote by $\mathscr{M}$ the cokernel of (1.1.3), then $\mathscr{M}$ becomes a left $\mathscr{D}_X$-module and $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is the kernel of (1.1.2). This means that the set of holomorphic solutions to $Pu = 0$ depends only on $\mathscr{M}$.

For this reason we mean by a system of linear differential equations a left $\mathscr{D}_X$-module.

1.2. Let us take a local coordinate system $(x_1, ..., x_n)$ of X. Then any differential operator P can be written in the form

$$P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x) \partial^\alpha \tag{1.2.1}$$

where $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}$, $|\alpha| = \alpha_1 + ... + \alpha_n$ and the $a_\alpha(x)$ are holomorphic functions. For $j \in \mathbf{N}$, we set

$$P_j(x, \xi) = \sum_{|\alpha| = j} a_\alpha(x) \xi^\alpha ,$$

where $\xi^\alpha = \xi_1^{\alpha_1} ... \xi_n^{\alpha_n}$, and we call $\{P_j(x, \xi)\}$ the *total symbol* of P. The largest m such that $P_m \neq 0$ is called the *order* of P and P_m is called the *principal symbol* of P and denoted by $\sigma(P)$.

Let us denote by T^*X the cotangent bundle of X, and let

$$(x_1, ..., x_n; \xi_1, ..., \xi_n)$$

be the associated coordinates of T^*X. It is a classical result that if we consider $\sigma(P)$ as a function on T^*X, then this does not depend on our choice of the local coordinate system $(x_1, ..., x_n)$.

1.3. Let M be a real analytic manifold, and X its complexification, e.g., $M = \mathbf{R}^n \subset X = \mathbf{C}^n$. Let P be a differential operator on X. When $\sigma(P)(x, \xi) \neq 0$ for $(x, \xi) \in \mathbf{R}^n \times (\mathbf{R}^n \backslash \{0\})$, P is called an elliptic differential operator. In this case, we have the following result.

PROPOSITION 1.3.1. *If u is a hyperfunction (or distribution) on M and Pu is real analytic, then u is real analytic. More precisely if we denote by $\mathscr{A}$ the sheaf of real analytic functions on M and by $\mathscr{B}$ (resp. $\mathscr{D}b$) the sheaf of hyperfunctions (resp. distributions) on M, then $P: \mathscr{B}/\mathscr{A} \to \mathscr{B}/\mathscr{A}$ (resp. $P: \mathscr{D}b/\mathscr{A} \to \mathscr{D}b/\mathscr{A}$) is a sheaf isomorphism.*

This suggests that if $\sigma(P)(x, \xi) \neq 0$, we can consider the inverse P^{-1} in a certain sense. Since (x, ξ) is a point of the cotangent bundle, P^{-1} is attached to the cotangent bundle.

In fact, as we shall see in the sequel, we can construct a sheaf of rings $\mathscr{E}_X$ on T^*X such that $\mathscr{D}_X \subset \pi_* \mathscr{E}_X$, where π is the canonical projection $T^*X \to X$. Moreover if $P \in \mathscr{D}_X$ satisfies $\sigma(P)(x, \xi) \neq 0$ at a point $(x, \xi) \in T^*X$, then P^{-1} exists as a section of $\mathscr{E}_X$ on a neighborhood of (x, ξ).

This can be compared to the analogous phenomena for polynomial rings, as shown in the following table.