# §0. Introduction

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# INTRODUCTION TO MICROLOCAL ANALYSIS [1])

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## § 0. INTRODUCTION

0.1. In this lecture, we explain the micro-local point of view (i.e. the consideration on the cotangent bundle) for the study of systems of linear differential equations.

0.2. The importance of the cotangent bundle in analysis has been recognized for a long time, although implicitly, for example by the following consideration.

We consider a linear differential operator

$$P(x, \partial) = \sum_{\alpha \in \mathbf{N}^n} a_\alpha(x) \partial^\alpha \quad \text{with} \quad \partial^\alpha = (\partial/\partial x_1)^{\alpha_1} \ldots (\partial/\partial x_n)^{\alpha_n}$$

for $\alpha = (\alpha_1, \ldots, \alpha_n)$, and try to find a solution to $P(x, \partial)u(x) = 0$. If we suppose that $u(x)$ has a singularity along a hypersurface $f(x) = 0$, then the simplest possible form of $u(x)$ is

$$u(x) = c_0(x) f(x)^s + c_1(x) f(x)^{s+1} + \ldots$$

Then setting $P_m(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi^\alpha$ we have

$$\begin{aligned} (0.1.1) \quad P(x, \partial)u(x) = {} & s(s-1) \ldots (s-m+1) c_0(x) P_m(x, df) f(x)^{s-m} + \ldots \\ & + (s+j) \ldots (s+j-m+1) c_j(x) P_m(x, df) \\ & + (\text{terms in } c_0, \ldots, c_{j-1}) f(x)^{s-j} + \ldots \end{aligned}$$

Therefore $P_m(x, df)$ must be a multiple of $f(x)$ (i.e. $P_m(x, df) = 0$ on $f^{-1}(0)$). In this case, $f^{-1}(0)$ is called characteristic.

Thus the hypersurface $f^{-1}(0)$ is not arbitrary and the singularity of the solution to $Pu(x) = 0$ has a very special form.

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[1]) Survey lectures given at the University of Bern in June 1984 under the sponsorship of the International Mathematical Union.

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0.3. If $P_m(x, \xi) \neq 0$ for any non-zero real vector $\xi$, then $P$ is called an elliptic operator. In this case, one can easily solve $P(x, \partial)u(x) = f(x)$ for any $f(x)$, at least locally. We start from the plane wave decomposition of the $\delta$-function.

$$\delta(x) = \text{const.} \int_{S^{n-1}} \frac{\omega(\xi)}{<x, \xi>^n} \tag{0.3.1}$$

where $\omega(\xi)$ is the invariant volume element of the sphere $S^{n-1}$.

By formula (0.1.1), we can solve

$$P(x, \partial)K(x, y) = \frac{1}{(<x, \xi> - <y, \xi>)^n},$$

by setting $K(x, y) = \sum c_j(<x, \xi> - <y, \xi>)^{m-n+j}$ and determining $c_j$ recursively. Then $K(x, y) = \text{const} \int K(x, y, \xi)\omega(\xi)$ satisfies

$$P(x, \partial)K(x, y) = \delta(x-y)$$

by (0.3.1).

If we set $u(x) = \int K(x, y)f(y)dy$ then $u(x)$ satisfies $P(x, \partial)u(x) = f(x)$. In fact

$$P(x, \partial)u(x) = \int P(x, \partial)K(x, y)f(y)dy = \int \delta(x-y)f(y)dy = f(x).$$

0.4. By these considerations, M. Sato recognized explicitly the importance of the cotangent bundle by introducing the singular spectrum of functions and microfunctions [Sato]. For a real analytic manifold $M$, let $\mathscr{A}_M$ be the sheaf of real analytic functions and $\mathscr{B}_M$ the sheaf of hyperfunctions. Let $\pi: T^*M \to M$ be the cotangent bundle of $M$. Then he constructed the sheaf $\mathscr{C}_M$ of microfunctions and an exact sequence

$$0 \to \mathscr{A}_M \to \mathscr{B}_M \overset{\text{sp}}{\to} \pi_* \mathscr{C}_M \to 0.$$

The action of a differential operator $P(x, \partial)$ on $\mathscr{B}_M$ extends to the action on $\mathscr{C}_M$.

Moreover $P: \mathscr{C}_M \to \mathscr{C}_M$ is an isomorphism outside

$$\{(x, \xi) \in T^*M; P_m(x, \xi) = 0\}.$$

0.5. In the situation of § 0.2, $u(x) = c_0(x)f(x)^s + \dots$ satisfies supp $\mathrm{sp}(u(x)) = \{\pm df(x)\}$. Therefore $P_m(x, df)$ must be zero if $P(x, \partial)u(x) = 0$. In fact otherwise the bijectivity of $P: \mathscr{C}_M \to \mathscr{C}_M$ implies $\mathrm{sp}(u) = 0$.

0.6. Such a method of studying functions or differential equations locally on the cotangent bundle is called microlocal analysis. After Sato's discovery of microfunctions, microlocal analysis was studied intensively in Sato-Kawai-Kashiwara [SKK].

Also L. Hörmander [H] worked in the $C^\infty$-case. Since then, microlocal analysis has been one of the most fundamental tools in the theory of linear partial differential equations.

## § 1. Systems of Differential Equations (See [O], [Bj])

1.1. Let $X$ be a complex manifold. A system of linear differential equations can be written in the form

(1.1.1) $$\sum_{j=1}^{N_0} P_{ij}(x, \partial)u_j = 0, \quad i = 1, 2, \dots, N_1 .$$

Here $u_1, \dots, u_{N_0}$ denote unknown functions and the $P_{ij}(x, \partial)$ are differential operators on $X$. The holomorphic function solutions of (1.1.1) are simply the kernel of the homomorphism

(1.1.2) $$P: \mathscr{O}_X^{N_0} \to \mathscr{O}_X^{N_1}$$

which assigns $(v_1, \dots, v_{N_1})$ to $(u_1, \dots, u_{N_0})$, where $v_i = \sum_j P_{ij}(x, \partial)u_j$.

Let us denote by $\mathscr{D}_X$ the ring of differential operators with holomorphic coefficients. Then

(1.1.3) $$P: \mathscr{D}_X^{N_1} \to \mathscr{D}_X^{N_0}$$

given by $(Q_1, \dots, Q_{N_1})$ to $(\Sigma Q_i P_{i1}, \dots, \Sigma Q_i P_{iN_0})$ is a left $\mathscr{D}_X$-linear homomorphism. If we denote by $\mathscr{M}$ the cokernel of (1.1.3), then $\mathscr{M}$ becomes a left $\mathscr{D}_X$-module and $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ is the kernel of (1.1.2). This means that the set of holomorphic solutions to $Pu = 0$ depends only on $\mathscr{M}$.

For this reason we mean by a system of linear differential equations a left $\mathscr{D}_X$-module.