

XI. Continuous operators are not always bounded

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COROLLARY 38. Assume that the definite space $(\mathfrak{E}; \langle , \rangle)$ is complete and that the system of types (Corollary 26) is linearly independent in $\Gamma/2\Gamma$ (considered as a $\mathbb{Z}_2$ -vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.

$C(\mathfrak{E})$ in Theorem 37 is not complete (unless finite dimensional). Its quadratic form $\langle , \rangle$ can be extended to the completion $\tilde{C}$. By using Theorem 28 one can see that this completion has $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$ if and only if E has $L_{\perp\perp}(E) = L_c(E)$.

XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let $\mathfrak{E}$ be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator) $h: \mathfrak{E} \rightarrow \mathfrak{E}$ is called *bounded* iff there exists $\gamma \in \Gamma$ such that for all $x \in \mathfrak{E}$ we have $\varphi\langle hx \rangle \geq \gamma + \varphi\langle x \rangle$.

In [6] A. Fässler gave an explicit example of a continuous operator h on an orthomodular space $\mathfrak{E}$ that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra $\mathcal{B}(\mathfrak{E})$ of bounded operators $h: \mathfrak{E} \rightarrow \mathfrak{E}$ when $\mathfrak{E}$ is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19) $(\mathfrak{E}; \langle , \rangle)$ contains a maximal orthogonal family $(e_i)_{\mathbb{N}}$ such that the groups $\Theta(\varphi\langle e_i \rangle)$ are different.

By (14) we see that (19) is a property of $\mathfrak{E}$, not of $(e_i)_{\mathbb{N}}$; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let $(\mathfrak{E}; \langle , \rangle)$ be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family $(e_i)_{\mathbb{N}}$ that enjoys (19). If $f: \mathfrak{E} \rightarrow \mathfrak{E}$ is given, expand (Lemma 27)

$$(20) \quad f e_i = \sum_{j \in \mathbb{N}} \alpha_{ij} e_j \quad (i \in \mathbb{N})$$

THEOREM 39 ([6]). The linear map f is bounded iff it is continuous and satisfies

$$(21) \quad \{\varphi\alpha_{ii} \mid T\varphi\langle fe_i \rangle = T\varphi\langle e_i \rangle\} \quad \text{is bounded below.}$$

The heart of the proof of Theorem 39 is the following consequence of assumption (19).

LEMMA 40 [6]. *If f is continuous then (19) implies that the set $I := \{i \in \mathbb{N} \mid \varphi\langle fe_i \rangle < \varphi\langle e_i \rangle \text{ \& } \varphi\langle fe_i \rangle \not\equiv \varphi\langle e_i \rangle \pmod{2\Gamma}\}$ is finite.*

Proof. We renumber the e_i such that $\Theta(\varphi\langle e_i \rangle) \subsetneq \Theta(\varphi\langle e_{i+1} \rangle)$. If we replace e_i by a multiple then its group does not change; therefore we may assume without loss of generality that for all $r, s \in \mathbb{N}$ we have

$$(22) \quad r < s \Rightarrow \varphi\langle e_r \rangle \in \Theta(\varphi\langle e_s \rangle), \quad \varphi\langle e_r \rangle \geq 0$$

From (22) we obtain that for all $r, s \in \mathbb{N}$

$$(23) \quad r < s \Rightarrow \forall \delta \in \Gamma: \varphi\langle e_r \rangle < |\varphi\langle e_s \rangle + 2\delta|$$

If $i \in I$ then $\varphi\langle fe_i \rangle \equiv \varphi\langle e_j \rangle$ for some $j \neq i$. Let $I_0 \subset I$ be the subset of those i for which the j is smaller than i . Thus, if $i \in I \setminus I_0$ then $\varphi\langle fe_i \rangle = \varphi\langle e_j \rangle + 2\varphi\alpha_{ij} < \varphi\langle e_i \rangle$; so by (23) we must actually have $\varphi\langle fe_i \rangle \leq -\varphi\langle e_i \rangle \leq 0$. Since (e_i) is a null sequence we see that $I \setminus I_0$ has to be finite (because $\{fe_i \mid i \in I \setminus I_0\}$ must also be a null sequence if $I \setminus I_0$ is infinite). Thus, in order to prove Lemma 40 we have to show that I_0 is finite.

The idea in [6] is to show that for each $i \in I_0$ there is $\lambda_i \in k$ such that $\varphi\langle f(\lambda_i e_i) \rangle \leq 0$ and $\varphi\langle \lambda_i e_i \rangle \geq 0$ so that by the same token I_0 must be finite. This is accomplished by choosing, in turn, $\lambda = 1$, $\lambda = \langle fe_i \rangle^{-1}$, according as to whether $\varphi\langle fe_i \rangle$ is ≤ 0 , > 0 respectively.

Proof of Theorem 39. Assume that f is bounded. Continuity is obvious. Let $\gamma \in \Gamma$ be a bound for f and let $\gamma_0 = \min\{0, \gamma\}$. Now $\varphi\langle fe_i \rangle = \varphi\langle \alpha_{ii} e_i \rangle$ for all i occurring in (21), i.e., for all $i \in \mathbb{N} \setminus I$ (by assumption (19) we have $T\varphi\langle e_i \rangle \neq T\varphi\langle e_j \rangle$ for all $i \neq j$). Thus, if $\varphi\alpha_{ii} > 0$ then trivially $\varphi\alpha_{ii} \geq \gamma_0$; if $\varphi\alpha_{ii} < 0$ then $\varphi(\alpha_{ii}) > 2\varphi\alpha_{ii} \geq \gamma \geq \gamma_0$.

Assume conversely that f is continuous and has (21). We show that there is $\gamma_0 \in \Gamma$ with $\varphi\langle fe_i \rangle \geq \gamma_0 + \varphi\langle e_i \rangle$ ($i \in \mathbb{N}$). Let γ be a lower bound for the set in (21) and set $\gamma_0 := \min\{0, 2\gamma, \gamma_1, \dots, \gamma_n\}$ where $\gamma_v := \varphi\langle fe_v \rangle - \varphi\langle e_v \rangle$, $v \in I$. To finish the proof we conclude $\varphi\langle f\mathbf{x} \rangle > \varphi\langle \mathbf{x} \rangle + \gamma_0$ ($\forall \mathbf{x}$) by continuity of f :

$$\varphi\langle f \sum_1^\infty \xi_i e_i \rangle = \varphi\langle f(\xi_{i_0} e_{i_0}) \rangle \geq \gamma_0 + \varphi\langle \xi_{i_0} e_{i_0} \rangle = \gamma_0 + \varphi\langle \mathbf{x} \rangle.$$