

V. Necessary conditions in D for $L_c = L_{\bot}$

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IV.3. THE CLASS $\mathcal{D}$ OF DEFINITE SPACES. Positive definite forms over ordered fields satisfy the triangle inequality as well as the Cauchy-Schwarz inequality. We therefore set down

Definition 15. A definite space is a nondegenerate hermitean space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ over an involutorial division ring $(k, *)$, $\text{char } k \neq 2$, that is equipped with a $*$ -valuation φ that has $\varphi(2) = 0$ (cf. Remark 35) and that satisfies one (and hence all) of the four statements in Lemma 14. A definite space $\mathfrak{E}$ will always be considered as a topological vector space, the topology being given by the zero-neighbourhood basis $\mathcal{U}_\gamma := \{\eta \in \mathfrak{E} \mid \varphi\langle \eta \rangle \geq \gamma\}$, $\gamma \in \Gamma$. If $(e_i)_{i \in I}$ is any family over vectors in $\mathfrak{E}$ such that the net of all finite ("partial") sums $\sum e_i$ has a limit x in $\mathfrak{E}$ then we write $x = \sum_{i \in I} e_i$ and call $(e_i)_{i \in I}$ summable.

LEMMA 16. Let $(e_i)_{i \in I}$ be an orthogonal family in the definite space $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ and $\mathfrak{F}$ its span. For each x in the topological closure of $\mathfrak{F}$ we have $x = \sum_{i \in I} \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i$.

Proof. Let $\mathcal{P}$ be the set of all finite subsets of I . For $V \in \mathcal{P}$ we set $x_V := \sum_{i \in V} \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i$. We have to prove that for each $\gamma \in \Gamma$ there is $U \in \mathcal{P}$ such that $\varphi\langle x - x_V \rangle \geq \gamma$ for all V with $U \subset V \in \mathcal{P}$. Now there is $\eta \in \mathfrak{F}$ with $\varphi\langle x - \eta \rangle \geq \gamma$. Pick $U \in \mathcal{P}$ with $\eta \in \text{span}\{e_i \mid i \in U\}$. If $U \subset V \in \mathcal{P}$ then $x - x_V \perp x_V - \eta$, so by "Pythagoras" (Lemma 14 (ii)) we obtain $\gamma \leq \varphi\langle x - \eta \rangle = \min\{\varphi\langle x - x_V \rangle, \varphi\langle x_V - \eta \rangle\} \leq \varphi\langle x - x_V \rangle$. $\square$

V. NECESSARY CONDITIONS IN $\mathcal{D}$ FOR $L_c = L_{\perp \perp}$

The principal result of this section is

THEOREM 17 ([20]). Let $\mathfrak{E}$ be an infinite dimensional definite space carrying an admissible topology i.e., the topology mentioned in Definition 15 is admissible in the sense of Definition 1; let furthermore $(e_i)_{i \in I}$ be an orthogonal family in $\mathfrak{E}$ such that $(\varphi\langle e_i \rangle)_{i \in I}$ has a lower bound in Γ . Then $\sum_{i \in I} e_i$ exists.

Proof. Let $\mathfrak{F} := \text{span}\{\langle e_i \rangle^{-1} e_i - \langle e_0 \rangle^{-1} e_0 \mid i \in I\}$. We first wish to show that $\langle e_0 \rangle^{-1} e_0$ is not an element of the topological closure $\overline{\mathfrak{F}}$. Indeed,

if γ is a lower bound of $(\varphi\langle e_i \rangle)_{i \in I}$ and if we let $x := \sum_{i \in U} \lambda_i (\langle e_i \rangle^{-1} e_i - \langle e_0 \rangle^{-1} e_0)$ be a typical vector of $\mathfrak{F}$ (U some finite nonvoid subset of $I \setminus \{0\}$) then we get the inequalities

$$\begin{aligned} \varphi\langle x - \langle e_0 \rangle^{-1} e_0 \rangle &= \varphi\langle (-1 - \sum_U \lambda_i) \langle e_0 \rangle^{-1} e_0 + \sum_U \lambda_i \langle e_i \rangle^{-1} e_i \rangle \\ &= \min_{i \in U} \{2\varphi(-1 - \sum_U \lambda_i) - \varphi\langle e_0 \rangle, 2\varphi(\lambda_i) - \varphi\langle e_i \rangle\} \\ &\leq 2 \min_{i \in U} \{\varphi(-1 - \sum_U \lambda_i), \varphi(\lambda_i)\} - \gamma \leq \varphi(-1) - \gamma = -\gamma. \end{aligned}$$

Thus $\overline{\mathfrak{F}} \neq \mathfrak{E}$.

Since $\mathfrak{F}^{\perp\perp} = \overline{\mathfrak{F}}$ we have $\mathfrak{F}^{\perp} \neq (0)$. Pick a non-zero $x \in \mathfrak{F}^{\perp}$; so $\langle x, e_i \rangle \langle e_i \rangle^{-1} = \langle x, e_0 \rangle \langle e_0 \rangle^{-1}$. If we assume that $(e_i)_{i \in I}$ is a maximal orthogonal family then by $L_c = L_s$ and Lemma 16 $x = \sum_I \langle x, e_i \rangle \langle e_i \rangle^{-1} e_i = \langle x, e_0 \rangle \langle e_0 \rangle^{-1} \sum_I e_i$ and thus $\sum_I e_i \in \mathfrak{F}^{\perp}$. If $(e_i)_{i \in I}$ is not maximal then we write it as a difference of two maximal bounded families: Complete the given family to a maximal orthogonal bounded family $(e_i)_{i \in J}$, $J \supset I$, by Zorn's Lemma. For $i \in J$ let $\alpha_i := 1 \in k$ when $i \in I$ and $\alpha_i := 2$ when $i \in J \setminus I$. The two families $(2e_i)_{i \in J}$, $(\alpha_i e_i)_{i \in J}$ are bounded maximal families to which the previous result may be applied. We get $\sum_{i \in I} e_i = \sum_{i \in I} (2e_i) - \sum_{i \in I} \alpha_i e_i \in \mathfrak{E}$. $\square$

COROLLARY 18. *If $\mathfrak{E}$ and $(e_i)_{i \in I}$ are as in Theorem 17 then $(e_i)_{i \in I}$ converges to $0 \in \mathfrak{E}$.* $\square$

COROLLARY 19. *If $\mathfrak{E}$ is as in Theorem 17 then the cofinality type of Γ is ω_0 . In particular, the topology on $\mathfrak{E}$ satisfies the first countability axiom.* $\square$

COROLLARY 20. *If $\mathfrak{E}$ is as in Theorem 17 then all orthogonal families of non-zero vectors are countable.*

Proof. Let $(e_i)_{i \in I}$ be such a family; by multiplying e_i by a suitable scalar, if necessary, we may assume $(\varphi\langle e_i \rangle)_{i \in I}$ to be bounded below. Since $\sum_{i \in I} e_i$ exists by Theorem 17, the sets $I_\gamma = \{i \in I \mid \varphi\langle e_i \rangle \leq \gamma\}$ are finite for all $\gamma \in \Gamma$. Let $(\gamma_i)_{i \in \mathbb{N}}$ be cofinal in Γ . Then $I = \cup \{I_{\gamma_i} \mid i \in \mathbb{N}\}$ is countable. $\square$

Definition 21. The elements of the group $\Gamma/2\Gamma$ are called *types*. Let $T: \Gamma \rightarrow \Gamma/2\Gamma$ be the canonical projection. $T \circ \varphi$ is constant on the square classes of k (elements of k/k^2) and $T \circ \varphi \circ \langle \rangle$ is constant on the "punctured"

straight lines in E . A family $(e_l)_{l \in I}$ of vectors in $\mathfrak{E}$ is said to satisfy the *type-condition* iff for all $(\alpha_l)_{l \in I} \in k^I$ the following holds: if $(\varphi\langle\alpha_l e_l\rangle)_{l \in I}$ is bounded (below) then $(\alpha_l e_l)_{l \in I}$ converges to $0 \in E$.

COROLLARY 22. Let $\mathfrak{E}$ be as in Theorem 17. $\Gamma/2\Gamma$ is infinite. Each orthogonal family in $\mathfrak{E}$ satisfies the type-condition, equivalently, $\Gamma/2\Gamma$ satisfies (8) below. $\square$

COROLLARY 23. Let $\mathfrak{E}$ be as in Theorem 17. Then k is complete.

Proof. By Corollary 19 it suffices to show that a sequence $(\alpha_i)_{i \in \mathbb{N}}$ with limit $0 \in k$ is summable. Let $(e_i)_{i \in \mathbb{N}}$ be maximal orthogonal in $\mathfrak{E}$ with $(\varphi\langle e_i\rangle)_{i \in \mathbb{N}}$ bounded below. If $(\lambda_i)_{i \in \mathbb{N}} \in k^{\mathbb{N}}$ has $(\varphi\langle\lambda_i\rangle)_{i \in \mathbb{N}}$ bounded below then $(\lambda_i e_i)_{i \in \mathbb{N}}$ is summable and by continuity of $\langle \cdot, \cdot \rangle$ we obtain

$$\langle \sum_{\mathbb{N}} \lambda_i e_i, \sum_{\mathbb{N}} e_i \rangle = \sum_{\mathbb{N}} \lambda_i \langle e_i \rangle.$$

Thus, all families $(\lambda_i \langle e_i \rangle)_{i \in \mathbb{N}}$ with bounded $(\lambda_i)_{i \in \mathbb{N}}$ are summable.

Pick a strictly monotonic sequence $(n_i)_{i \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ with $u_0 = 0$ and for all $i \in \mathbb{N}^+$ and all $m \geq n_i$: $\varphi(\alpha_m) > \varphi\langle e_i \rangle$, and set $A_i := \sum \{\alpha_j \mid n_i \leq j < n_{i+1}\}$. The family $(A_i)_{i \in \mathbb{N}}$ is summable if and only if $(\alpha_i)_{i \in \mathbb{N}}$ summable and, if the sums exist, these must be equal. If we set $\lambda_i := A_i \langle e_i \rangle^{-1}$ then, by what we have shown, the family of the $A_i = \lambda_i \langle e_i \rangle$ is summable. $\square$

COROLLARY 24. Let $\mathfrak{E}$ be as in Theorem 17. Then $\mathfrak{E}$ is complete.

Proof. Let $(x_i)_{i \in \mathbb{N}}$ be a Cauchy sequence (Corollary 19). For each fixed $\eta \in \mathfrak{E}$ the map $x \mapsto \langle \eta, x \rangle$ is uniformly continuous. Hence by Cor. 23 the map $f: \eta \mapsto \lim_i \langle \eta, x_i \rangle$ is well-defined. As it is a continuous linear map, its kernel is a closed hyper-plane and so $(L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}))$ there is $a \in \mathfrak{E}$ such that $f(\eta) = \langle x, a \rangle$. Let $N \subseteq \mathbb{N}$ be infinite. Because $\lim \varphi\langle \eta, a - x_i \rangle = \infty$ for all $\eta \in \mathfrak{E}$ it follows by systematic use of the Cauchy-Schwarz inequality that $\{\varphi\langle a - x_i \rangle \mid i \in N\}$ is not bounded above by any $\gamma \in \Gamma$. Therefore $(x_i)_{i \in \mathbb{N}}$ converges to a .

VI. SUFFICIENT CONDITIONS IN $\mathscr{D}$ FOR $L_c = L_{\perp\perp}$

VI.1. ASSUMPTIONS. In this chapter $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$ is a definite space in the sense of Definition 15. Of the base field k we shall furthermore assume