

## 2. Relative completeness and co-completeness

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **02.05.2024**

### **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

### **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

PROPOSITION 1.5. *Let  $Q$  be perfect and  $K$  hypoabelian. An extension  $K \rightarrow G \xrightarrow{\pi} Q$  lies in the image of  $A$  if and only if both*

- a)  $\pi$  is an epimorphism preserving perfect radicals (that is,  $\pi \mathcal{P}G = Q$ ); and
- b)  $[\mathcal{P}G, K] = 1$ .

These conditions are easily verified for an extension where the kernel lies in the hypercentre of  $G$ . For then  $K$  must be nilpotent, so that condition (a) is satisfied by [3 (2.3) (iii)]. On the other hand, because  $G$  acts nilpotently on  $K$  so does  $\mathcal{P}G$ ; by Kaluzhnin's theorem [18 (7.1.1.1)] the image of the perfect group  $\mathcal{P}G$  in  $\text{Aut}(K)$  induced by conjugation is nilpotent and hence trivial. This result also admits a converse, for if  $K$  is nilpotent then the extension obtained by the construction  $A$  is easily seen to have its kernel in the hypercentre.

COROLLARY 1.6. *Let  $K$  be a nilpotent group and  $Q$  perfect. Then the set of equivalence classes of extensions with kernel  $K$  in the hypercentre and with quotient  $Q$  is in 1 – 1 correspondence with  $H^2(Q; Z(K)) \cong \text{Hom}(H_2(Q), Z(K))$ .*

Here  $H_2(Q) = H_2(Q; \mathbf{Z})$  is just the Schur multiplier of  $Q$ . The given isomorphism is immediate from the universal coefficient theorem because  $Q$  is perfect.

## 2. RELATIVE COMPLETENESS AND CO-COMPLETENESS

This paragraph takes us to the point of departure for our examples.

PROPOSITION 2.1. *Suppose that groups  $Q$  and  $K$  have the property that every homomorphism from  $Q$  to  $\text{Out}(K)$  is trivial. Then every extension with kernel  $K$  and quotient  $Q$  is trivial, provided that also either*

- (a)  $K$  is centreless, or
- (b)  $Q$  is superperfect.

Case (a) of (2.1) is of course known and includes the example of complete groups (that is,  $\text{Out}(K) = 1$  too). It follows immediately from (1.3).

Case (b) requires a little more attention. Recall that  $Q$  superperfect means that its first and second homology groups with trivial integer coefficients both vanish. By means of the universal coefficient/Künneth formulae, the first and second cohomology with arbitrary trivial coefficients are also zero. So both the  $H^2$  and  $\text{Hom}$  sets in the exact sequence of (1.1) are singletons.

The following terminology is suggested by (2.1). Let  $\mathbf{Q}$  be a class of groups. Then  $K$  is *complete relative to  $\mathbf{Q}$*  if every extension with kernel  $K$  and quotient in  $\mathbf{Q}$  is trivial. The next result is widely known, but is included for the sake of completeness (sic).

**PROPOSITION 2.2.** *A group  $K$  is complete if and only if it is complete relative to all groups.*

*Proof.* It remains to establish the sufficiency of the relative condition. We first show that  $K$  is centreless. This can be done by a little homological algebra applied to  $H^2$  in the exact sequence of (1.1). More directly, let  $Z_1$  and  $Z_2$  be two copies of  $Z = Z(K)$ , equipped with isomorphisms  $\theta_j: Z \rightarrow Z_j$ ,  $j = 1, 2$ . In the group  $K \times (Z_1 * Z_2)$ , let  $\bar{Z}$  denote the normal closure of the subgroup generated by all elements of the form

$$(z, 1) [(1, \theta_1(z)), (1, \theta_2(z))]$$

with  $z \in Z$ , and let  $G$  be the quotient  $(K \times (Z_1 * Z_2)) / \bar{Z}$ . Evidently  $K$  is normal in  $K \times (Z_1 * Z_2)$  and so in  $G$ . From the triviality of the extension  $K \xrightarrow{\iota} G \twoheadrightarrow G/\iota(K)$ , there is a left inverse  $\rho: G \rightarrow K$  to  $\iota$ . Thus the triviality of each  $[\iota K, (1, \theta_j(z))]$  in  $G$  implies that of  $[K, \rho(1, \theta_j(z))]$  in  $K$ , making each  $\rho(1, \theta_j(z)) \in Z$ . Then any  $z \in Z$  satisfies

$$\begin{aligned} z = \rho(z) &= \rho(z, 1) = \rho[(1, \theta_2(z)), (1, \theta_1(z))] \\ &= [\rho(1, \theta_2(z)), \rho(1, \theta_1(z))] \\ &\in [Z, Z] = 1. \end{aligned}$$

Hence  $K$  is indeed centreless, so that (1.3) applies. In particular the set  $\text{Hom}(\text{Out}(K), \text{Out}(K))$  must be a singleton, forcing  $\text{Out}(K) = 1$ .

The literature contains various results which may be expressed as examples of relative completeness (such as [21 Exercises 536, 537]). It is sometimes convenient to dualise this phraseology. Thus, for a class  $\mathbf{C}$  of groups, we say that  $Q$  is *co-complete relative to  $\mathbf{C}$*  if  $\mathcal{E}\text{xt}(Q, K)$  is trivial whenever  $K \in \mathbf{C}$ . The asymmetry between these two concepts is highlighted by the absence of a counterpart to (2.2); that is, there are no non-trivial (absolutely) co-complete groups. To see this, consider the left regular representation of  $Q$  regarded as a non-trivial homomorphism from  $Q$  to the automorphism group of the free abelian group  $\text{Fr}(Q)_{ab}$  generated by the elements of  $Q$ . The semi-direct product which results (via  $E$ ) is then a non-trivial element of  $\mathcal{E}\text{xt}(Q, \text{Fr}(Q)_{ab})$ .

This example is quite suggestive inasmuch as, in order to find a group relative to which the quotient  $Q$  is not co-complete, we have passed to a group which is large in comparison with  $Q$ . One might therefore speculate on the existence of quotient groups which are co-complete relative to all groups of a certain size. Examples of such quotients are presented in the next paragraph.

### 3. EXAMPLES

In view of (2.1), our examples are of superperfect groups  $Q$  whose homomorphic images of sufficiently small cardinality, say  $\leq \alpha$ , are all trivial. For this purpose it is worth recalling that an abelian group with a generating set of cardinality  $\beta$  has automorphism group of order at most  $2^\beta$ . We feature three types of example.

#### I. *The acyclic groups considered by de la Harpe and McDuff*

Acyclic groups have the same homology (with trivial integer coefficients) as the trivial group and so are certainly superperfect. On the other hand, the acyclic groups discussed in [12] have the further property that any countable image is trivial. Hence they are *co-complete relative to all  $K$  with  $\text{Out}(K)$  countable*, and in particular relative to all finitely generated groups.

#### II. *The universal central extension over a simple group*

Let  $S$  be a non-abelian simple group. Being perfect,  $S$  admits a universal central extension  $Q$  [14], [17] (that is, an initial object in the category of all equivalence classes of extensions with central kernel and quotient  $S$ ). Now  $Q$  is well-known to be superperfect — indeed, it is the unique superperfect central extension over  $S$  —, so we consider its possible images.

**PROPOSITION 3.1.** *The non-trivial homomorphic images of  $Q$  are precisely the perfect central extensions of  $S$ .*

Since any image of  $Q$  is also perfect, clearly not all central extensions over  $S$  need be obtained in this way. For example, take the direct product of such an extension with an abelian group. However, if  $E$  has quotient  $S$  and central kernel then by [2 (1.6)b)] so does its maximal perfect subgroup  $\mathcal{P}E$ . So every central extension contains a preferred perfect central