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PROPOSITION 1.5. *Let Q be perfect and K hypoabelian. An extension $K \rightarrowtail G \overset{\pi}{\twoheadrightarrow} Q$ lies in the image of* A *if and only if both*

a) π *is an epimorphism preserving perfect radicals (that is, $\pi\mathscr{P}G = Q$); and*

b) $[\mathscr{P}G, K] = 1$.

These conditions are easily verified for an extension where the kernel lies in the hypercentre of G. For then K must be nilpotent, so that condition (a) is satisfied by [3 (2.3) (iii)]. On the other hand, because G acts nilpotently on K so does $\mathscr{P}G$; by Kaluzhnin's theorem [18 (7.1.1.1)] the image of the perfect group $\mathscr{P}G$ in $\mathrm{Aut}(K)$ induced by conjugation is nilpotent and hence trivial. This result also admits a converse, for if K is nilpotent then the extension obtained by the construction A is easily seen to have its kernel in the hypercentre.

COROLLARY 1.6. *Let K be a nilpotent group and Q perfect. Then the set of equivalence classes of extensions with kernel K in the hypercentre and with quotient Q is in $1-1$ correspondence with $H^2(Q; Z(K)) \cong \mathrm{Hom}(H_2(Q), Z(K))$.*

Here $H_2(Q) = H_2(Q; \mathbf{Z})$ is just the Schur multiplier of Q. The given isomorphism is immediate from the universal coefficient theorem because Q is perfect.

2. RELATIVE COMPLETENESS AND CO-COMPLETENESS

This paragraph takes us to the point of departure for our examples.

PROPOSITION 2.1. *Suppose that groups Q and K have the property that every homomorphism from Q to $\mathrm{Out}(K)$ is trivial. Then every extension with kernel K and quotient Q is trivial, provided that also either*

(a) *K is centreless, or*

(b) *Q is superperfect.*

Case (a) of (2.1) is of course known and includes the example of complete groups (that is, $\mathrm{Out}(K) = 1$ too). It follows immediately from (1.3).

Case (b) requires a little more attention. Recall that Q superperfect means that its first and second homology groups with trivial integer coefficients both vanish. By means of the universal coefficient/Künneth formulae, the first and second cohomology with arbitrary trivial coefficients are also zero. So both the H^2 and Hom sets in the exact sequence of (1.1) are singletons.

The following terminology is suggested by (2.1). Let $\mathbf{Q}$ be a class of groups. Then K is *complete relative to* $\mathbf{Q}$ if every extension with kernel K and quotient in $\mathbf{Q}$ is trivial. The next result is widely known, but is included for the sake of completeness (sic).

PROPOSITION 2.2. *A group K is complete if and only if it is complete relative to all groups.*

Proof. It remains to establish the sufficiency of the relative condition. We first show that K is centreless. This can be done by a little homological algebra applied to H^2 in the exact sequence of (1.1). More directly, let Z_1 and Z_2 be two copies of $Z = Z(K)$, equipped with isomorphisms $\theta_j \colon Z \to Z_j$, $j = 1, 2$. In the group $K \times (Z_1 * Z_2)$, let $\overline{Z}$ denote the normal closure of the subgroup generated by all elements of the form

$$(z, 1)\,[(1, \theta_1(z)), (1, \theta_2(z))]$$

with $z \in Z$, and let G be the quotient $(K \times (Z_1 * Z_2))/\overline{Z}$. Evidently K is normal in $K \times (Z_1 * Z_2)$ and so in G. From the triviality of the extension $K \overset{\iota}{\rightarrowtail} G \twoheadrightarrow G/\iota(K)$, there is a left inverse $\rho \colon G \to K$ to ι. Thus the triviality of each $[\iota K, (1, \theta_j(z))]$ in G implies that of $[K, \rho(1, \theta_j(z))]$ in K, making each $\rho(1, \theta_j(z)) \in Z$. Then any $z \in Z$ satisfies

$$\begin{aligned} z = \rho\iota(z) = \rho(z, 1) &= \rho[(1, \theta_2(z)), (1, \theta_1(z)] \\ &= [\rho(1, \theta_2(z)), \rho(1, \theta_1(z))] \\ &\in [Z, Z] = 1 . \end{aligned}$$

Hence K is indeed centreless, so that (1.3) applies. In particular the set $\mathrm{Hom}(\mathrm{Out}(K), \mathrm{Out}(K))$ must be a singleton, forcing $\mathrm{Out}(K) = 1$.

The literature contains various results which may be expressed as examples of relative completeness (such as [21 Exercises 536, 537]). It is sometimes convenient to dualise this phraseology. Thus, for a class $\mathbf{C}$ of groups, we say that Q is *co-complete relative to* $\mathbf{C}$ if $\mathscr{E}\mathrm{xt}(Q, K)$ is trivial whenever $K \in \mathbf{C}$. The asymmetry between these two concepts is highlighted by the absence of a counterpart to (2.2); that is, there are no non-trivial (absolutely) co-complete groups. To see this, consider the left regular representation of Q regarded as a non-trivial homomorphism from Q to the automorphism group of the free abelian group $\mathrm{Fr}(Q)_{ab}$ generated by the elements of Q. The semi-direct product which results (via E) is then a non-trivial element of $\mathscr{E}\mathrm{xt}(Q, \mathrm{Fr}(Q)_{ab})$.

This example is quite suggestive inasmuch as, in order to find a group relative to which the quotient Q is not co-complete, we have passed to a group which is large in comparison with Q. One might therefore speculate on the existence of quotient groups which are co-complete relative to all groups of a certain size. Examples of such quotients are presented in the next paragraph.

3. Examples

In view of (2.1), our examples are of superperfect groups Q whose homomorphic images of sufficiently small cardinality, say $\leqslant \alpha$, are all trivial. For this purpose it is worth recalling that an abelian group with a generating set of cardinality β has automorphism group of order at most 2^{β}. We feature three types of example.

I. *The acyclic groups considered by de la Harpe and McDuff*

Acyclic groups have the same homology (with trivial integer coefficients) as the trivial group and so are certainly superperfect. On the other hand, the acyclic groups discussed in [12] have the further property that any countable image is trivial. Hence they are *co-complete relative to all* K *with* $\operatorname{Out}(K)$ *countable*, and in particular relative to all finitely generated groups.

II. *The universal central extension over a simple group*

Let S be a non-abelian simple group. Being perfect, S admits a universal central extension Q [14], [17] (that is, an initial object in the category of all equivalence classes of extensions with central kernel and quotient S). Now Q is well-known to be superperfect — indeed, it is the unique superperfect central extension over S —, so we consider its possible images.

Proposition 3.1. *The non-trivial homomorphic images of* Q *are precisely the perfect central extensions of* S.

Since any image of Q is also perfect, clearly not all central extensions over S need be obtained in this way. For example, take the direct product of such an extension with an abelian group. However, if E has quotient S and central kernel then by [2 (1.6)b)] so does its maximal perfect subgroup $\mathscr{P}E$. So every central extension contains a preferred perfect central