

§3. Z/4-MANIFOLDS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **03.05.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$$\begin{array}{ccccccc}
H^2(\mathbf{Z}/p^2; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p^2; M) & \rightarrow & 0 \\
\downarrow \Delta = i_*(\Lambda_1) & & \downarrow i_*(M) & & \\
H^2(\mathbf{Z}/p; \varphi_2 \Lambda_2) & \xrightarrow{\lambda_*^k} & H^2(\mathbf{Z}/p; \Lambda_1) & \rightarrow & H^2(\mathbf{Z}/p; M) & \rightarrow & 0
\end{array}$$

where Δ is the diagonal map $\mathbf{Z}_p \rightarrow p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\text{Im}(\Delta) \subset \text{Im}(\lambda_*^k)$. Let e denote a column p -vector consisting of all 1's, according to the proof of (2.5) we must find an $\bar{x}_k$, $1 \leq k \leq p-1$ so that $C_{p,k} \cdot \bar{x}_k = C_{p,1}^k \cdot \bar{x}_k = e$. We do this inductively

on k . For example, $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we

define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1. \end{cases}$$

Clearly $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$, for all coordinates except possibly the first; we must show $\bar{x}_k(p) \equiv 0 \pmod{p}$. But a comparison of the $\bar{x}_k$'s with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1} \binom{1}{0} \equiv 0 \pmod{p},$$

since $k-1 < p-1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3. $\mathbf{Z}/4$ -MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0)$, $i = 6, 8, 9$. According to (2.7), the indecomposable $\mathbf{Z}/4$ -lattices that carry special classes are M_1 , M_4 and M_9 . It is easy to see M_i is faithful if and only if

$i = 3, 5, 6, 7, 8, 9$. Hence if $M = \sum m_i M_i$ is an arbitrary $\mathbf{Z}/4$ -lattice then M is a faithful representation carrying a special class if and only if the multiplicities m_i satisfy the inequalities:

$$m_1 + m_4 + m_9 > 0$$

(3.0)

$$m_3 + m_5 + m_6 + m_7 + m_8 + m_9 > 0.$$

Since the multiplicities are a complete set of isomorphism invariants in the case $p = 2$ (see section 1) one can use the conditions (3.0) to show:

(3.1) THEOREM. If $L_n(m)$ denotes the number of isomorphism classes of n -dimensional $\mathbf{Z}/m$ -lattices that carry special classes, then:

$$L_n(4) = \sum_{j=2}^{n-1} \left(a_j - \left[\frac{j}{2} \right] - 1 \right) + \sum_{j=[n]_2+2}^{n-2} (a_j - a_{j-1} - 1) \\ + \sum_{j=[n]_4}^{n-4} (a_j - a_{j-2} - a_{j-4} + a_{j-6})$$

where $[k]_p$ denotes the reduction of k modulo p , $[k]$ denote the largest integer $\leq k$ and the a_j 's are given by

$$P(t) = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^3}$$

In particular, the number of n -dimensional $\mathbf{Z}/4$ -manifolds is at least $L_n(4)$.

Proof. If $Q(t)$ is a power series, let $\text{coef}(n, Q(t))$ denote the coefficient of t^n in $Q(t)$. The number $L_n(4)$ counts the number of ways of writing

$$n = m_1 + m_2 + 2(m_3 + m_4) + 3(m_5 + m_8) + 4(m_6 + m_7 + m_9)$$

where the m_i 's satisfy (3.0). If $m_1 > 0$ there is a contribution:

$$\sum_{m_1=1}^{n-2} \text{coef}(n-m_1, P(t)) - \left(\left[\frac{n-m_1}{2} \right] + 1 \right)$$

where $\left[\frac{n-m_1}{2} \right] + 1$ is the number of ways of expressing $n - m_1$ as a combination of 1's (M_2) and 2's (M_4) (not permitted by (3.0)). Reindexing gives the first term for $L_n(4)$.

Similarly, if $m_1 = 0, m_4 > 0$ there is a contribution:

$$\sum_{m_4} \text{coef}(t^{n-2m_4}(1-t)P(t)) - 1$$

where 1 is subtracted to omit choosing m_2 alone. Finally, if $m_1 = m_4 = 0$, we have:

$$\sum_{m_9} \text{coef}(t^{n-4m_9}, (1-t^2)(1-t^4)P(t)).$$

The coefficients of $(1-t)P(t)$ and $(1-t^2-t^4-t^6)P(t)$ are easily expressible in terms of the a_j 's and the result follows.

Remark. In order for a $\mathbf{Z}/p$ -lattice to carry a special class, the multiplicity of the trivial representation must be non-zero. Topologically this is reflected in the fact that a $\mathbf{Z}/p$ -manifold fibers over a circle. This is already false for a 4-dimensional $\mathbf{Z}/4$ -manifold as the following example shows.

Example. $L_4(4) = 6$. The multiplicities of the indecomposables in these 4-dimensional $\mathbf{Z}/4$ -lattices are given by:

notation of [2]	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
07/02/02	1				1				
12/01/04	1							1	
12/01/02	1	1	1						
07/02/01	2		1						
12/01/03			1	1					
12/01/06									1

where the first column gives the label of these " $\mathbf{Z}$ -classes" from the table of the four-dimensional crystallographic groups in [2]. In fact, as these tables indicate, there is precisely one $\mathbf{Z}/4$ -manifold corresponding to each $\mathbf{Z}/4$ -lattice, hence there are exactly 6 4-dimensional $\mathbf{Z}/4$ -manifolds.

Remark. Recall that if $p < 23$, the field $\mathbf{Q}(e^{2\pi i/p})$ has class number one. This fact, along with the work of Charlap [4], shows that the number of n -dimensional $\mathbf{Z}/p$ -manifolds is exactly $L_n(p)$, $p < 23$. This number is readily computable, as Charlap [4, p. 30] remarks, and the precise formula is:

$$(3.2) \quad L_n(p) = \sum_{j=p-1}^{n-1} \left(\left[\frac{j}{p-1} \right] - \left\langle \frac{j}{p} \right\rangle + 1 \right)$$

where $\langle k \rangle$ denotes the smallest integer $\geq k$. In particular, $L_p(p) = 1$, $L_n(p) = 0$, $p > n$, and when $p = 2$

$$(3.3) \quad L_n(2) = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right) - 1 + \frac{[n]_2}{4}$$

using the notation of (3.1).

One can easily construct the following table of values of $L_n(p)$:

n	2	3	4	5	6
p					
2	1	3	5	8	11
3		1	2	3	4
5				1	2

Hence 14 of the 74 4-dimensional flat manifolds have cyclic holonomy ≤ 5 . (Furthermore, 26 have holonomy the Klein 4-group.) We describe analogous facts in dimension 5 below.

We let $SH^2(H, M)$ denote the set of special classes in $H^2(H, M)$. If H is a cyclic p -group and $i: \mathbb{Z}_p \hookrightarrow H$ is the inclusion of the subgroup of order p , then

$$SH^2(H, M) = H^2(H, M) - \ker(i^*).$$

If N (resp. Z) denotes the normalizer (resp. the centralizer) of H in $\text{Aut}(M)$, there is an exact sequence (see [15, p. 50])

$$0 \rightarrow Z \rightarrow N \rightarrow \text{Aut}(H).$$

We conjecture:

Conjecture. If $\mathbb{Z}[e^{2\pi i/p^k}]$ is a unique factorization domain for, $1 \leq k \leq n$, then N acts transitively on $SH^2(\mathbb{Z}/p^n; M)$ for any H -lattice M .

The case $n = 1$ of the Conjecture follows from Charlap [4]. Class number tables shows that the $n = 2$ case applies to $p = 2, 3, 5$, the $n = 3$ case to 2, 3 and the $n = 4$ case to $p = 2$. This conjecture implies that the lower bound of (3.1) is exact.

We mention that the multiplicities of the indecomposables in the 5-dimensional $\mathbb{Z}/4$ -lattices that admit special classes are given by:

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
	1								1
	1						1		
*						1			
	1	1			1				
*	1	1						1	
	1		1	1					
*	1		2						
*	1	2	1						
*	3							1	
	2				1				
*	2	1	1						
	3		1						
*		1	1	1					
				1	1				
*				1				1	
*		1							1

Those lattices that are starred clearly satisfy the Conjecture.

REFERENCES

- [1] BERMAN, S. and P. GUDIKOV. Indecomposable representations of finite groups over the ring of p -adic integers. *Amer. Math. Soc. Trans., Series 2*, vol. 50 (1966), 77-113.
- [2] BROWN, H., R. BÜLOW, J. NEUBÜSER, H. WONDRATSCHEK and H. ZASSENHAUS. *Crystallographic groups of four-dimensional space*. John Wiley, New York, 1978.
- [3] BROWN, K. *Cohomology of Groups*. Springer Verlag, New York, 1983.
- [4] CHARLAP, L. Compact flat Riemannian manifolds I. *Ann. Math.* 81 (1965), 15-30.
- [5] CURTIS, C. W. and I. REINER. *Representation theory of finite groups and associative algebras*. Interscience, New York, 1966.
- [6] DIEDERICHSEN, F. Über die Ausreduktion ganzzahliger Guppendarstellungen bei arithmetischer Äquivalenz. *Abh. Hans. Univ.* 13 (1938), 357-412.
- [7] HELLER, A. and I. REINER. Representations of cyclic groups in rings of integers, I, II. *Ann. Math. (2)* 76 (1962), 73-92 and 77 (1963), 318-328.
- [8] HILLER, H. Minimal dimension of flat manifolds with abelian holonomy. *To appear*.