§1. Genera of Z/p^2-lattices

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The case $H = \mathbf{Z}/p^2$ differs fundamentally from that of $H = \mathbf{Z}/p$ in that the number of genera of indecomposable $\mathbf{Z}/p^2$-lattices is a function of p (in fact, $4p+1$) while there are 3 genera of indecomposable $\mathbf{Z}/p$-lattices, for any prime p. Furthermore it is no longer the case that only the trivial representation admits non-trivial (and "special") two-dimensional cohomology classes. Eventually we restrict to the case $p = 2$ where there are 9 genera (originally described by Roiter [14] correcting a mistake in Diederichsen [6]) and 3 of these admit special classes. The assumption $p = 2$ also insures that the genera are identical to the isomorphism classes so there are no further invariants to consider. This follows from work of Reiner [13] and the fact that $\mathbf{Z}[e^{2\pi i/m}]$ is a unique factorization domain for $m = 2, 4$.

As the smallest dimension of a $\mathbf{Z}/p^2$-manifold is $p^2 - p + 1$ (this is a special case of results from [8]) only the case $p = 2$ produces flat manifolds of dimension 5, the smallest dimension for which one lacks a complete classification. We show that there are at least 16 5-dimensional flat manifolds with holonomy $\mathbf{Z}/4$ and give a general lower bound for any dimension.

The main ingredient for these results is the work of Heller and Reiner [7] on the integral representation theory of $\mathbf{Z}/p^2$, reviewed in section 1. In section 2 we study the cohomology of the indecomposables, compute their restrictions to the subgroup of order p and identify the "special" classes. Finally in section 3 we restrict to the case $p = 2$ and study the class of $\mathbf{Z}/4$-manifolds.

It is hoped that this example of holonomy classification will succeed in exposing the role played by integral representation theory and cohomology of groups in understanding the structure of flat Riemannian manifolds.

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§ 1. Genera of $\mathbf{Z}/p^2$-lattices

We begin by briefly reviewing the language and philosophy of the integral representation theory of finite groups (see [5], [11]). We then give Heller and Reiner's description [7] of the genera of $\mathbf{Z}/p^2$ lattices as extensions.

Suppose Λ is a $\mathbf{Z}$-order in a $\mathbf{Q}$-algebra A. A Λ-*lattice* is a left Λ-module that is also a free abelian group of finite rank. The basic problem of integral representation theory is the classification of such Λ-lattices, Λ fixed. The $\mathbf{Z}$-orders that we will need are group rings of finite groups $ZG \subset \mathbf{Q}G$ and rings of algebraic integers $\mathcal{O}_K$ in an algebraic number field K. We sometimes refer to a $\mathbf{Z}G$-lattice as a G-lattice.

Let $\mathbf{Z}_p$ (resp. $\mathbf{Q}_p$) denote the p-adic completion of $\mathbf{Z}$ (resp. $\mathbf{Q}$). It is easy to see that $\Lambda_p = \mathbf{Z}_p \otimes \Lambda$ is a $\mathbf{Z}_p$-order in the $\mathbf{Q}_p$-algebra A_p. Furthermore, any Λ-lattice M yields a Λ_p-lattice $M_p = \mathbf{Z}_p \otimes M$. One says that M and M' are *locally isomorphic* (or in the same *genus*) if $M_p \cong M'_p$ as Λ_p-modules for all primes p.

The classification of Λ-lattices is often attacked by a "local-to-global" approach. By this we mean the solution of the following two problems:

1. (local) Determine a complete set of invariants of the genus of a Λ-lattice.

2. (global) Determine a complete set of invariants of the isomorphism class of Λ-lattice within a fixed genus.

This approach has been very successful and the examples below illustrate it.

We introduce some notation. Let ω (resp. ζ) denote a primitive p^2 (resp. p^{th}) root of unity and let $R_1 = \mathbf{Z}[\zeta]$, $R_2 = \mathbf{Z}[\omega]$. We also let $\Lambda_i = \mathbf{Z}[\mathbf{Z}/p^i]$. The classification of lattices over a ring of algebraic integers, or more generally a Dedekind domain is classic, and is a good example of the local-to-global approach.

(1.1) THEOREM (Steinitz). *If R is a Dedekind domain, every R-lattice is a direct sum of non-zero ideals of R. The genus of an R-lattice is determined by the number of non-zero ideals occurring, its* rank. *The isomorphism class of the R-lattice within the genus is determined by the ideal class of the product of the ideals as an element of the ideal class group of R (the* Steinitz class *of the lattice).*

The classification problem for $\mathbf{Z}/p$-lattices was solved by Diederichsen [6] and Reiner [10]. Again the local-to-global approach is useful.

If $\mathfrak{a}$ denotes a non-zero ideal of R_1 let $E(\mathfrak{a})$ denote the non-split extension of $\mathfrak{a}$ by the trivial lattice $\mathbf{Z}$. The genus of $\mathfrak{a}$ (resp. $E(\mathfrak{a})$) is denoted α (resp. β). Every $\mathbf{Z}/p$-lattice M can be written:

$$M = \mathbf{Z}^{(a)} \oplus \sum_{i=1}^{b} \mathfrak{a}_i \oplus \sum_{i=1}^{c} E(\mathfrak{a}'_i)\,.$$

The genus of the $\mathbf{Z}/p$-lattice M is determined by the multiplicities a, b, c of the three indecomposable genera $\mathbf{1}$, α, β. The isomorphism class of the lattice within its genus is completely determined by the ideal class $\prod_{i=1}^{b} \mathfrak{a}_i \cdot \prod_{i=1}^{c} \mathfrak{a}'_i$ in the ideal class group of R_1.

The solution of the (local) classification problem for $R_2 = \mathbf{Z}[\omega]$ (a Dedekind domain) and $\Lambda_1 = \mathbf{Z}[\mathbf{Z}/p]$ can be combined to classify genera of $\mathbf{Z}/p^2$-lattices. The technique used is essentially homological. If M is a $\mathbf{Z}/p^2$-lattice, we let $L = \{x \in M : (x^p - 1)M = 0\}$. L is a $\mathbf{Z}/p$-lattice and fits into a $\mathbf{Z}/p^2$-exact sequence

$$0 \to L \to M \to N \to 0$$

where N is an R_2-lattice. Hence one is reduced to classifying extensions of R_2-lattices by $\mathbf{Z}/p$-lattices using homological methods. It is not difficult to show (see [13, p. 478]) that $\mathrm{Ext}_2(R_2, L) \equiv L/pL$, where L is an arbitrary $\mathbf{Z}/p$-lattice. In fact, if $\alpha \in L/pL$ then the corresponding extension is given by the pushout diagram:

$$\begin{array}{ccccccccc} 0 & \to & \varphi_2\Lambda_2 & \to & \Lambda_2 & \to & R_2 & \to & 0 \\ & & {}^{\alpha}\downarrow & & \downarrow & & \| & & \\ 0 & \to & L & \to & M & \to & R_2 & \to & 0 \end{array} \tag{1.2}$$

where φ_2 is the cyclotomic polynomial $1 + x^p + x^{2p} + \dots + x^{(p-1)p}$ and the map α is (by abuse of language) the map that sends φ_2 to α. We write (L, α) for this extension. The final result is:

(1.3) THEOREM (Heller and Reiner [7]). *There are* $4p + 1$ *genera of indecomposable* $\mathbf{Z}/p^2$*-lattices given by:*

$$\begin{aligned} M_1 &= \mathbf{Z}, \\ M_2 &= R_1, \\ M_3 &= R_2, \\ M_4 &= \Lambda_1, \\ M_5 &= (\mathbf{Z}, 1), \\ M_6(k) &= (\Lambda_1, \lambda^k), && 0 \leqslant k \leqslant p-1, \\ M_7(k) &= (Z \oplus \Lambda_1, \mathbf{1} \oplus \lambda^k), && 1 \leqslant k \leqslant p-2, \\ M_8(k) &= (R_1, \lambda^k), && 0 \leqslant k \leqslant p-2, \\ M_9(k) &= (Z \oplus R_1, \mathbf{1} \oplus \lambda^k), && 0 \leqslant k \leqslant p-2, \end{aligned}$$

where $\lambda = (1-x)$, $\Lambda_1 \cong \mathbf{Z}[x]/(x^p - 1)$ *and we view* R_1 *as a quotient of* Λ_1.

The splintering of these genera into isomorphism classes has been analyzed by Reiner [13]. One can, of course, replace R_1, R_2 by ideal classes in these rings and Λ_1 by $E(\mathfrak{a})$ (cf. section 1) where $\mathfrak{a}$ is an ideal class in R_1. There is an additional invariant lying in a quotient of the group of units of a certain finite ring and, if $p \equiv 1 \pmod 4$ a certain quadratic residue character mod p can also appear as an invariant. The precise result is Theorem 7.3 of [13]. We will require only the observation [13, p. 494] that if $p = 2, 3$ there are no further invariants, i.e. each genus of an indecomposable is a single isomorphism class. In the case $p = 5$ already, although the class number of $\mathbf{Q}(e^{2\pi i/m})$ is one for $m = 5, 25$, the 21 genera of indecomposables split up into 40 isomorphism classes. Hence already the further isomorphism invariants mentioned above exert an influence.

§ 2. Cohomology, restrictions and special classes

If H is a finite group, M an H-lattice then: $H^i(H, M) \cong \oplus_p H^i(H, M_p)$, where p ranges over the primes dividing the order of H [3, p. 84]. Hence if M and M' are locally isomorphic, $H^i(H, M) \cong H^i(H, M')$; so the cohomology of an H-lattice depends only on its genus.

We recall the cohomology of a cyclic group $\mathbf{Z}/n = \langle\sigma\rangle$ [3, p. 58]. We write $N = 1 + \sigma \ldots + \sigma^{n-1}$ and $D = 1 - \sigma$. If M is a $\mathbf{Z}/n$-module, then

$$H^0(\mathbf{Z}/n, M) = M^\sigma$$
$$H^{2i-1}(\mathbf{Z}/n, M) = {}_NM/D \cdot M$$
$$H^{2i}(\mathbf{Z}/n, M) = M^\sigma/N \cdot M$$

for all $i \geqslant 1$, where M^σ denotes σ-invariants and ${}_NM = \{x \in M : Nx = 0\}$. From these remarks it is easy to compute the cohomology of the indecomposable $\mathbf{Z}/p$-lattices described in section 1.

(2.1) Proposition. *The following table describes the cohomology of the indecomposable $\mathbf{Z}/p$-lattices:*

M	rank	H^0	H^1	H^2
$\mathbf{1}$	1	$\mathbf{Z}$	0	$\mathbf{Z}/p$
α	$p-1$	0	$\mathbf{Z}/p$	0
β	p	$\mathbf{Z}$	0	0