Appendix B

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Appendix B

Let K be a separable quadratic extension of a field k. We denote $x \mapsto \bar{x}$ the non trivial element $\mathrm{Gal}(K/k)$. Let D be a simple algebra with dimension n^2 over its center K. We will check the criterion of the text, for the existence of an involution of the second kind on D, i.e. of an anti-involution * of D, inducing $x \mapsto \bar{x}$ on K. The criterion is that $N_{K/k}\ \mathrm{cl}(D) = 0$ in $\mathrm{Br}(k)$.

Let us localize, for the étale topology, over $\mathrm{Spec}(k)$. This means making large enough étale extensions of scalars $\otimes_k k'$, and keeping track of the functoriality in k'. The field K becomes the separable quadratic extension $K' = K \otimes_k k'$ of k'. The algebra D becomes $D' = D \otimes_k k'$, and is of the form $D' = \mathrm{End}_{K'}(V')$, for V' a free module K'. The module V' is not determined uniquely by D', only up to homotheties (the corresponding projective space is uniquely determined).

For any K-module M, let M^- be the module deduced from M by the extension of scalars $^- : K \to K$, i.e. the module, unique up to unique isomorphism, provided with an anti-linear isomorphism $x \mapsto \bar{x} : M \xrightarrow{\sim} M^-$. Similarly for K'-modules. If $D' = \mathrm{End}(V')$, then $D^{-\prime} = \mathrm{End}(V'^-)$, and

$$(D \otimes_K D^-)' = \mathrm{End}(V' \otimes V'^-)\,.$$

Let W' be the fixed subspace of the anti-linear automorphism of $V' \otimes V'^-$ defined by $v \otimes \bar{w} \mapsto w \otimes \bar{v}$. It is the space of Hermitian forms on the dual of V'. One has $W' \otimes_{k'} K' = V' \otimes V'^-$. If $D_1 \subset D \otimes_K D^-$ is the fixed subspace of the anti-linear automorphism of $D \otimes_K D^-$ defined by $x \otimes \bar{y} \mapsto y \otimes \bar{x}$, then D'_1 is the k'-form of the K'-algebra $(D \otimes_K D^-)' = \mathrm{End}(V' \otimes V'^-)$ deduced from the k'-form W' of the K'-module $V' \otimes V'^- : D'_1 = \mathrm{End}_{k'}(W')$.

Involutions of the second kind on D' correspond one to one to non degenerate Hermitian forms on V', taken up to a factor (in k'^*). Those, in turn, by the "dual form" construction, correspond to "non degenerate" elements of W'. Again, one has to take them up to a factor. The projective space $\mathbf{P}(W')$ over k' is determined up to unique isomorphism by D'. It is hence (this is the point of localisation) defined over $k : \mathbf{P}(W') = P \otimes_k k'$, functorially in k'. The k-points of P (rather, the non degenerate points) parametrize the involutions of the second kind on D.

The functorial isomorphism $D'_1 = \mathrm{End}_{k'}(W')$ shows that P is the form of projective space (Severi-Brauer variety) attached to D_1. It has a rational point, and is then the ordinary projective space, if and only if D_1 is a matrix algebra.

This shows that D has involutions of the second kind if and only if the class of D_1 in $\mathrm{Br}(k)$ is trivial. This class is the required norm $N_{K/k}(\mathrm{cl}(D))$. In the localization spirit, this can be deduced from the fact that the homothety by $\lambda \in K'^*$ of V' induces on W' the homothety by $N_{K'/k'}(\lambda) \in k'^*$.

Appendix C

For $n \geqslant 3$, examples can be obtained as follows: take $k' = \mathbf{Q}[\zeta]$, with $\zeta = \exp(2\pi i/n)$, and $k = k' \cap \mathbf{R}$. Fix $a, b \in k^*$ and let D be the k'-algebra generated by X, Y, subject to

$$X^n = a\,, \qquad Y^n = b$$
$$XY = \zeta YX\,.$$

It admits the anti-involution $*$, inducing complex conjugation on k', defined by $\zeta^* = \zeta^{-1}$, $X^* = X$, $Y^* = Y$. The algebra D is of the type we require, provided it is a division algebra. This happens already with $a, b \in \mathbf{Z}$: take for a a prime congruent to 1 mod n, and for b an integer whose residue mod a has in the cyclic group of order n $(\mathbf{Z}/(a))^*/(\mathbf{Z}/(a))^{*n}$ an image of exact order n. For instance $n = 3$, $a = 7$, $b = 2$. For $n = 2$, one proceeds similarly with $k' = \mathbf{Q}[i]$, $\zeta = -1$, a congruent to 1 mod 4 and b not a square mod a. For instance, $a = 5$, and $b = 2$. In each case, the assumption on a ensures that k' embed in the a-adic completion $\mathbf{Q}_a$ of $\mathbf{Q}$, and the fact that D is a division algebra can be seen locally at a : $D \otimes_{k'} \mathbf{Q}_a$ is a division algebra with center $\mathbf{Q}_a$.