6. Young's rule, the specialization order and nilpotent matrices

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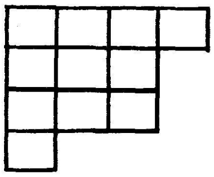
6.1. *Young Diagrams and Semistandard Tableaux.* Let $\kappa = (\kappa_1, ..., \kappa_m)$ be a partition of n. As usual we picture κ as a Young diagram; that is an array of n boxes arranged in m rows with κ_i boxes in row i, as in the following example

(6.2) $\kappa = (4, 3, 3, 2)$



Let $\lambda = (\lambda_1, ..., \lambda_s)$ be another partition of n. Then a semistandard κ-tableaux of type λ is the Young diagram of κ with the boxes labelled by the integers $1, ..., s$ such that i occurs λ_i times, $i = 1, ..., s$ and such that the labels are nondecreasing in each row of the diagram and strictly increasing along each column. An example of a $(5, 3, 2)$-tableaux of type $(4, 2, 2, 2)$ is

(6.3)
$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 4 \\ 2 & 2 & 3 & & \\ 3 & 4 & & & \end{array}$$

We shall use $K(\kappa, \lambda)$ to denote the number of different semistandard κ-tableaux of type λ; these numbers are sometimes called Kostka numbers.

6.4. *Young's Rule.* Let $[\rho]$ denote the irreducible representation associated to the partition ρ. Then Young's rule (cf. [29]) says that

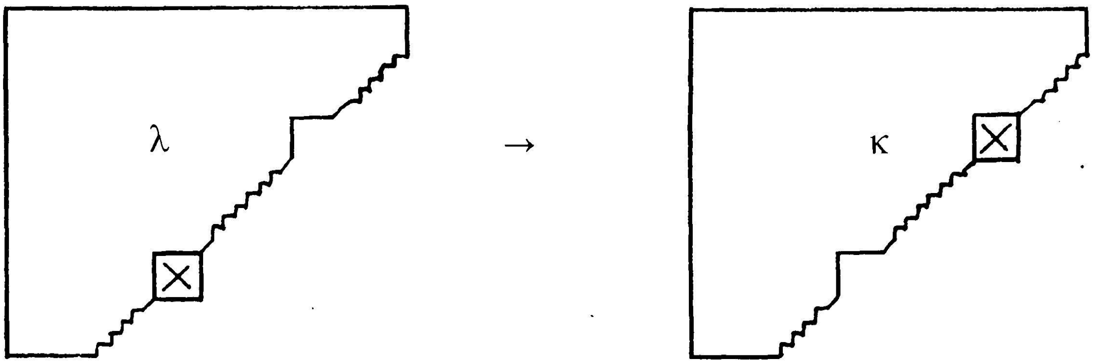
6.5. *Theorem.* Let κ and λ be partitions of n. Then the number of times that the irreducible representation $[\lambda]$ occurs in the permutation representation $\rho(\kappa)$ is equal to the number $K(\lambda, \kappa)$ of semistandard λ-tableaux of type κ.

6.6. *The Specialization order and Semistandard Tableaux.* The implication $\kappa > \lambda \leftarrow \rho(\lambda)$ is a direct summand of $\rho(\kappa)$ follows easily from this. First, however, we state a lemma which is standard and seemingly unavoidable when dealing with the specialization order. Its proof is easy.

6.7. *Lemma.* Let $\lambda = (\lambda_1, ..., \lambda_m)$ and $\kappa = (\kappa_1, ..., \kappa_m)$ be two partitions of n and suppose that $\lambda > \kappa$ and $(\lambda > \mu > \kappa) \Rightarrow (\lambda = \mu$ or $\mu = \kappa)$ for all partitions μ. Then there are an i and a j, $i < j$ such that

$$\kappa_i = \lambda_i + 1, \lambda_i < \lambda_{i-1}, \kappa_j = \lambda_j - 1, \lambda_j > \lambda_{j+1}, \kappa_s = \lambda_s, s \neq i, j\,.$$

Pictorially the situation looks as follows



That is a box in row j which can be removed without upsetting $\#(\text{row } j) \geqslant \#(\text{row } j+1)$ (which means that we must have had $\lambda_j > \lambda_{j+1}$) is moved to a higher row i which is such that it can receive it without upsetting $\#(\text{row } i) \leqslant \#(\text{row } i-1)$ (which means that we must have had $\lambda_i < \lambda_{i-1}$). We will say that λ covers κ. Of course not all transformations of the type described above result in a pair λ, κ such that there is no μ strictly between λ and κ.

6.8. *Lemma.* Let λ and κ be two partitions of n and suppose that there exists a semistandard λ-tableaux of type κ. Then $\kappa > \lambda$.

Proof. In a semistandard λ-tableaux of type κ all labels i must occur in the first rows (because the labels in the columns must be strictly increasing). The number of labels j with $j \leqslant i$ is $\kappa_1 + \ldots + \kappa_i$ and the number of places available in the first i rows is $\lambda_1 + \ldots + \lambda_i$. Hence

$$\lambda_1 + \ldots + \lambda_i \geqslant \kappa_1 + \ldots + \kappa_i$$

for all i so that $\lambda < \kappa$.

6.9. *The Implication* $[\kappa]$ *occurs in* $\rho(\lambda) \Rightarrow \kappa < \lambda$. Now suppose that $[\kappa]$ occurs in $\rho(\lambda)$. Then there is a semistandard κ-tableaux of type λ by Young's rule so that $\kappa < \lambda$ by lemma 6.8.

This implies, of course, that: $\rho(\kappa)$ is a subrepresentation of $\rho(\lambda) \rightarrow (\kappa < \lambda)$. Because there is obviously a semistandard κ- tableaux of type κ (in fact precisely one).

6.10. *The Implication* $\kappa < \lambda \Rightarrow \rho(\kappa)$ *is a subrepresentation of* $\rho(\lambda)$. To obtain this implication it suffices by Young's rule to show that the Kostka numbers satisfy $K(\mu, \kappa) \leqslant K(\mu, \lambda)$ if $\kappa < \lambda$ for all μ. To see this it is convenient to define $K(\mu, \nu)$ as the number of semistandard μ-tableaux of type ν for any sequence of nonnegative integers $\nu = (\nu_1, ..., \nu_s)$ such that $|\nu| = n$. Let $\bar{\nu} = (\bar{\nu}_1, ..., \bar{\nu}_s)$ denote the rearrangement of the ν_i such that $\bar{\nu}_1 \geqslant \bar{\nu}_2 \geqslant ... \geqslant \bar{\nu}_s$. Then $K(\mu, \nu) = K(\mu, \bar{\nu})$ and from this (non trivial) fact combined with lemma 6.7 it is easy to see that $K(\mu, \kappa) \leqslant K(\mu, \lambda)$ if $\kappa < \lambda$. (Assume λ covers κ and rearrange both so that the two changing entries are the first two.) We owe these remarks (indirectly) to A. Lascoux.

6.11. *Nilpotent Matrices and Representations* [11]. Let N_κ be the set of nilpotent matrices labelled by the partition κ, cf. 2.11 above. Let $\bar{N}_\kappa$ be its closure and let C be the set of diagonal matrices. Now take the scheme theoretic intersection of the closed subvarieties $\bar{N}_\kappa$ and C of the scheme of $n \times n$ matrices over **C**. This is a finite **C**-algebra with an obvious S_n-action. This turns out to be the permutation representation $\rho(\kappa)$ and using results from [39] a proof of the Snapper, Liebler-Vitale, Lam, Young theorem can be deduced. One very nice thing about this construction is that it also makes sense for the other classical simple Lie algebras and their Weyl groups. There are also relations with the so-called Springer representations of Weyl groups, [40-42].

7. Nilpotent matrices and systems

As was remarked in section 5 above the connection A in the diagram above essentially consists of an almost identical proof of the two theorems. We start with a proof of the Gerstenhaber-Hesselink theorem. The first ingredient which we shall also need for the feedback orbits theorem is the following elementary remark on ranks of matrices.

7.1. *Lemma.* Let $A(t)$ be a family of matrices depending polynomially on a complex or real parameter t. Suppose that rank $A(t) \leqslant$ rank $A(t_0)$ for all t. Then rank $A(t) =$ rank $A(t_0)$ for all but finitely many t.

This follows immediately from the fact that a polynomial in t has only finitely many zeros.

Let A be a nilpotent matrix. Then of course the similarity type of A is determined by the sequence of numbers.

$$n_i = \dim \operatorname{Ker} A^i .$$

The numbers $e_i = n_{i+1} - n_i$ form a partition of n and are dual to the partition formed by the sizes of the Jordan blocks.