5. Interrelations

where the last * in each row is nonzero. The closure of this open Schubert-cell is the Schubert-cell $SC(\gamma)$ defined in (4.3) above.

One easily checks that

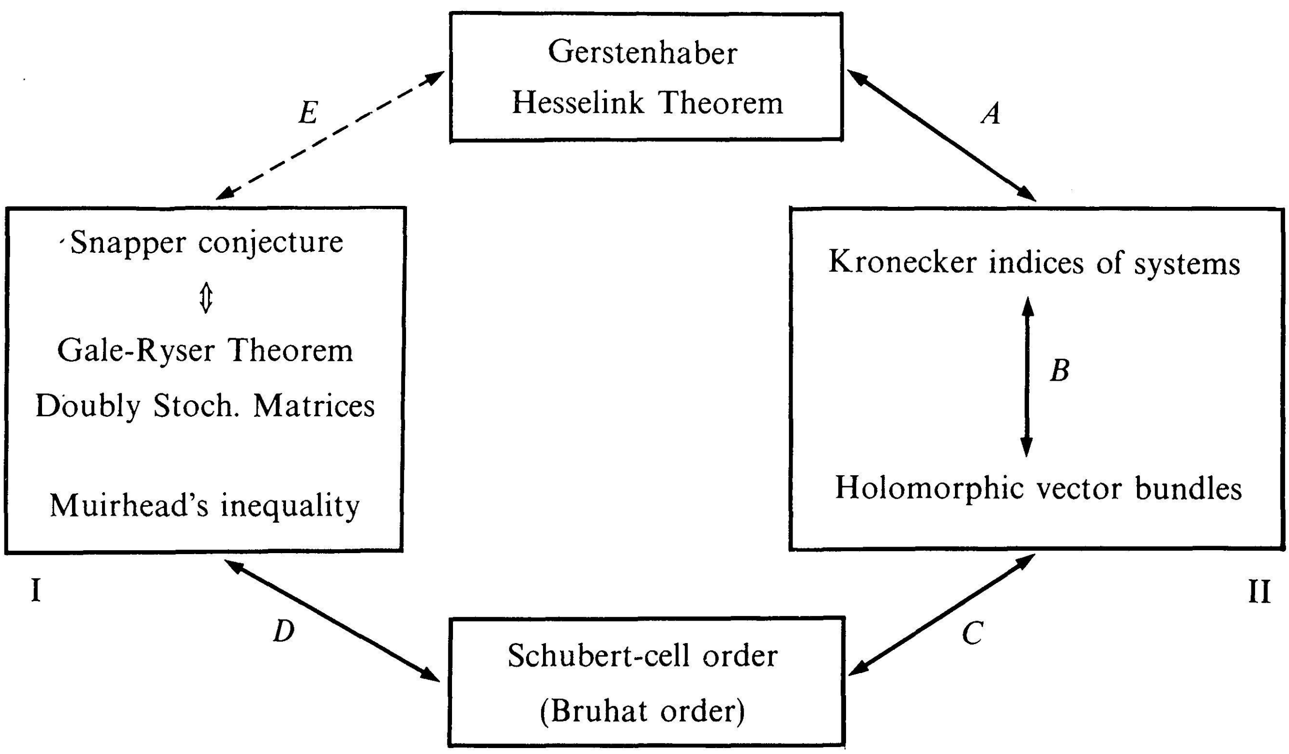
$$SC(\mu) \subset SC(\gamma) \leftrightarrow \mu_i \leqslant \gamma_i,\ i = 1, \ldots, n \tag{4.8}$$

and this order on the Schubert cells $SC(\gamma)$, or the equivalent ordering on n-tuples of natural numbers, is therefore a quotient of the Bruhat order on the Weyl group S_{n+m}. It is the induced order on the set of cosets $(S_n \times S_m)\sigma$, $\sigma \in S_{n+m}$. (Obviously if $\tau \in S_n \times S_m$, then $\tau\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$ if $\sigma(e_{\gamma_i}) \in \{e_1, \ldots, e_n\}$.) (And inversely the Bruhat order is determined by the associated orders of Schubert cells in the sense that $\sigma > \tau$ in S_n iff for all $k = 1, \ldots, n-1$ we have for the associated Schubert cells in $\mathbf{G}_k(\mathbf{C}^n)$ that $SC(\sigma) \supset SC(\tau)$; this is a rather efficient way of calculating the Bruhat order on the Weyl group S_n.)

5. Interrelations

Now that we have defined the concepts we need we can start to describe some interrelations between the various manifestations of the specialization order we discussed in section 2 above.

5.1. *Overview of the Various Relations.* A schematic overview of the various interconnections is given by the following diagram. In this diagram we



have put together in boxes the manifestations which are more or less known to be intimately related and have explicitly indicated the new relations to be discussed in detail below.

5.2. *On the various Relations.* The manifestations of the specialization order in box I are wellknown to be intimately related [2, 5, 10, 12, 18]. In particular, cf. [5] for the relations between doubly stochastic matrices, Muirheads inequality and the specialization order, which brings in also the marriage theorem and the Birkhoff-v. Neumann theorem that every doubly stochastic matrix is a convex linear combination of permutation matrices. For the relations of the Gale-Ryser theorem with the more or less combinatorial entities just mentioned cf. [12, 18] and also [2] which also contains lattice theoretic information on the partially ordered set of partitions with the specialization order.

Besides the Snapper conjecture (i.e. the Snapper, Liebler-Vitale, Lam, Young theorem) the Ruch-Schönhofer theorem [17], cf. also [20] also belongs in box I. This theorem states that $< \rho(\kappa), \bar{\rho}(\mu) > \; = 1$ if and only if $\kappa > \mu^*$ where $< \, , >$ denotes the usual inner product (which counts how many irreducible representations there are in common), and where $\bar{\rho}(\mu)$ is the representation of S_n obtained by inducing up the alternating representation of the Young subgroup S_μ. One way to link this theorem with the Gale-Ryser theorem is via Mackay's intertwining number theorem [10, 28] and Coleman's characterization [27] of double cosets of Young subgroups, cf. [10]. Another way goes via a beautiful formula of Snapper which we now explain (in a somewhat simplified case). Let $X = \{1, 2, ..., n\}$ with S_n acting on it in the natural way. Let Y be a finite set. A *weight* on Y is simply a function $w : Y \to \mathbf{N} \cup \{0\}$. Given a function $f : X \to Y$ its weight $w(f)$ is defined by $w(f)(y) = \# f^{-1}(y)$, where $\#$ denotes cardinality. For each weight w on Y let

$$I(w) = \{f : X \to Y \mid w(f) = w\} .$$

Now S_n acts on Y^X the space of functions from X to Y by $\sigma(f)(x) = f(\sigma^{-1}(x))$ and $I(w)$ is obviously invariant under this action. This associates a permutation representation $\rho(w)$ with each weight w on Y. Now consider two finite sets Y_1 and Y_2 with weights w_1 and w_2. Let $Y_1 \times Y_2$ be the product and π_1, π_2 the natural projections on Y_1 and Y_2. Define $M(w_1, w_2)$ as the set of all weights w on $Y_1 \times Y_2$ such that $w_i(y_i) = w(\pi_i^{-1}(y_i))$ for all $y_i \in Y_i$, $i = 1, 2$. Finally let $\underline{M}(w_1, w_2)$ be the sum of the characters belonging to the weights $w \in M(w_1, w_2)$. Then Snapper's formula says

$$< \underline{M}(w_1, w_2), \chi > \; = \; < \rho(w_1)\rho(w_2), \chi > \tag{5.3}$$

for all characters χ. To connect this result with statements on integrals matrices, it remains to note that $< \underline{M}(w_1, \cdot w_2), 1 >$ is the number of integral matrices with row sums w_1 and column sums w_2 and to prove that $< \underline{M}(w_1, w_2), \delta >$ is the number of (0, 1)-matrices with row sums w_1 and column sums w_2. Here δ is the alternating character of S_n.

Relation A in the diagram is essentially established by giving two virtually identical (but dual) proofs of the theorems, and these results can then be used to give natural continuous isomorphisms between feedback orbits of systems and similarity orbits of nilpotent matrices. More details are in section 7 below. For connection B one associates to a system $\Sigma \in L^{cr}_{m,n}$ a vector bundle $E(\Sigma)$ of dimension m over $\mathbf{P}^1(\mathbf{C})$. The construction used is a modification of the one in [14], cf. section 8 below. It has the advantage that one sees immediately that $\kappa(\Sigma) = \kappa(E(\Sigma))$. For connection C one uses the classifying morphism $\Psi_E : \mathbf{P}^1(\mathbf{C}) \rightarrow \mathbf{G}_n(\mathbf{C}^{n+m})$ attached to a positive bundle E over $\mathbf{P}^1(\mathbf{C})$ (cf. section 3.2 above). It turns out that the invariants of E can be recovered from Ψ_E by considering the dimensions of the spaces $A_1, \ldots, A_n$ such that $Im\Psi_E \subset SC(\underline{A})$, cf. section 9 below. To establish a link between representations of S_{n+m} and Schubert-cells we construct a family of representations of S_{n+m} parametrized by $\mathbf{G}_n(\mathbf{C}^{n+m})$, which can be used to give a deformation type proof of the Snapper conjecture (in the Liebler-Vitale form) (cf. section 12 below). This is not the shortest proof but it contains in it a purely elementary proof which uses no representations theory at all [6]. Combining the links A, C, D gives of course a link from the Gerstenhaber-Hesselink theorem to the Snapper conjecture, albeit a tenuous one. However, there is also a very direct link, due to Kraft [11], cf. section 6 below, and this gives yet another proof of the Snapper conjecture.

One possible approach to the Snapper conjecture is, of course, via Young's rule (discussed below in section 6), which states that the irreducible representation $[\kappa]$ occurs in $\rho(\lambda)$ with a multiplicity equal to the number of semistandard κ-tableaux of type λ. This can be made the basis of a proof and gives yet another link between the Snapper, Liebler-Vitale, Lam, Young theorem and the Gerstenhaber-Hesselink theorem. Both can be seen as consequences of the statement that there exists a semistandard λ-tableau of type μ iff $\lambda < \mu$, cf. section 7.6 below.

Finally let us remark that the proof of the increasing mixing character theorem for thermodynamic processes of Ruch and Mead follows readily from the theorem about doubly stochastic matrices described in 2.3 above.