2. SEVERAL MANIFESTATIONS OF THE SPECIALIZATION ORDER

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **29 (1983)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **28.04.2024**

2. Several manifestations of the specialization order

A schematic overview of the various relations of the specialization order to be described below can be found in section 5 of this paper.

2.1. *The Snapper, Liebler-Vitale, Lam, Young theorem (formerly the Snapper conjecture).* Let S_n be the group of permutations on n letters. Let $\kappa = (\kappa_1, \ldots, \kappa_m)$ be a partition of n and let S_κ be the corresponding Young subgroup $S_\kappa = S_{\kappa_1} \times \ldots \times S_{\kappa_m}$, where S_{κ_i} is seen as the subgroup of S_n acting on the letters $\kappa_1 + \ldots + \kappa_{i-1} + 1, \ldots, \kappa_1 + \ldots + \kappa_i$. (If $\kappa_m = 0$ the factor S_{κ_m} is deleted). Take the trivial representation of S_k and induce this up to S_n. Let $\rho(\kappa)$ denote the resulting induced representation. It is of dimension $\binom{n}{\kappa} = n!/\kappa_1! \ldots \kappa_m!$ and it can be easily described as follows. Take m symbols $a_1, \ldots, a_m$ and consider all associative (but non-commutative) words $\varepsilon_1 \ldots \varepsilon_n$ of length n in the symbols $a_1, \ldots, a_m$ such that a_i occurs precisely κ_i times. Let $W(\kappa_1, \ldots, \kappa_m) = W(\kappa)$ denote this set, then S_n acts on $W(\kappa)$ by $\sigma^{-1}(\varepsilon_1 \ldots \varepsilon_n) = \varepsilon_{\sigma(1)}\varepsilon_{\sigma(2)} \ldots \varepsilon_{\sigma(n)}$. Let $V(\kappa)$ be the vector space with the elements of $W(\kappa)$ as basis vectors. Extending the action of S_n linearly to $V(\kappa)$ gives a representation of S_n and this representation is $\rho(\kappa)$.

Now the irreducible representations of S_n are also labelled by partitions. Let $[\kappa]$ be the irreducible representation belonging to the partition κ. Snapper [20] proved that $[\kappa]$ occurs in $\rho(\kappa')$ only if $\kappa < \kappa'$ and conjectured the reverse implication. Liebler and Vitale [13] proved that $\kappa < \kappa'$ implies that $\rho(\kappa)$ is a direct summand of $\rho(\kappa')$ which, of course, implies that $\kappa < \kappa'$ which in turn implies that $[\kappa]$ occurs in $\rho(\kappa')$. Another proof of the implication (via a different generalization) is given in Lam [12]. Still another proof can be based on Young's rule, cf. section 6 below, and a completely elementary proof can be found in [6]. It is probably correct to ascribe the result in the first place to Young.

2.2. *The Gale-Ryser Theorem* ([18]). Let μ and ν be two partitions of n. Then there is a matrix consisting of zeros and ones whose columns sum to μ and whose rows sum to ν iff $\nu > \mu^*$. Here μ^* is the dual partition of μ defined by $\mu_i^* = \# \{j \mid \mu_j \geqslant i\}$. For example, $(2, 2, 1)^* = (3, 2)$.

2.3. *Doubly Stochastic Matrices.* A matrix $M = (m_{ij})$ is called doubly stochastic if $m_{ij} \geqslant 0$ for all i, j and if all the columns and all the rows add up to 1. Let μ and ν be two partitions of n. One says that μ is an average of ν if there is a doubly stochastic matrix M such that $\mu = M\nu$. Then there is the theorem that μ is an average of ν iff $\mu > \nu$ in the specialization order.

2.4. *Muirhead's Inequality.* One of the best-known inequalities is

$$(x_1 \cdot \ldots \cdot x_n)^{1/n} \leqslant n^{-1}(x_1 + \ldots + x_n) .$$

A far-reaching generalization due to Muirhead [21] goes as follows. Given a vector $p = (p_1, \ldots, p_n)$, $p_i \geqslant 0$, one defines a symmetrical mean (of the nonnegative variables $x_1, \ldots, x_n$) by the formula

$$[p]\,(x) = (n!)^{-1} \sum_{\sigma} x_1^{p_{\sigma(1)}} \ldots x_n^{p_{\sigma(n)}} \tag{2.5}$$

where the sum runs over all permutations $\sigma \in S_n$. Then one has Muirhead's inequality which states that $[p]\,(x) \leqslant [q]\,(x)$ for all nonnegative values of the variables $x_1, \ldots, x_n$ iff p is an average of q, so that in case p and q are partitions of n this happens iff $p > q$. The geometric mean-arithmetic mean inequality thus arises from the specialization relation $(1, \ldots, 1) > (n, 0, \ldots, 0)$.

2.6. *Completely Reachable Systems.* Let $L_{m,\,n}$ denote the space of all pairs of real matrices (A, B) of sizes $n \times n$ and $n \times m$ respectively. To each pair (A, B) one associates a control system given by the differential equations

$$\dot{x} = Ax + Bu,\ x \in \mathbf{R}^n,\ u \in \mathbf{R}^m \tag{2.7}$$

where the u's are the inputs or controls. The pair (A, B), or equivalently, the system (2.7), is said to be *completely reachable* if the reachability matrix $R(A, B) = (B\ AB \ldots A^nB)$ consisting of the $n+1$ $(n \times m)$-blocks A^iB, $i = 0, \ldots, n$, has maximal rank n. In system theoretic terms this is equivalent to the property that for any two points $x, x' \in \mathbf{R}^n$ one can steer $x(t)$ to x' in finite time starting from $x(0) = x$ by means of suitable control functions $u(t)$.

Let $L^{cr}_{m,\,n}$ denote the space of all completely reachable pairs of matrices (A, B). The Lie-group F of all block lower diagonal matrices $\begin{pmatrix} S & 0 \\ K & T \end{pmatrix}$, $S \in \mathbf{GL}_n(\mathbf{R})$, $T \in \mathbf{GL}_m(\mathbf{R})$, K an $m \times n$ matrix, acts on $L^{cr}_{m,\,n}$ according to the formula

$$(A, B)^g = (SAS^{-1} + SBTS^{-1}K,\ SBT),\ g = \begin{pmatrix} S & 0 \\ K & T \end{pmatrix} \tag{2.8}$$

The "generating transformations" $(A, B) \to (SAS^{-1}, SB)$ (base change in state space), $(A, B) \to (A, BT^{-1})$ (base change in input space) and $(A, B) \to (A + BK, B)$ (state space feedback), occur naturally in design problems (of control loops) in electrical engineering. It is a theorem of Brunovsky [30] and Kalman [9] and Wonham and Morse [31] that the orbits of F acting on $L^{cr}_{m,\,n}$ correspond bijectively with partitions of n. The partition belonging to $(A, B) \in L^{cr}_{m,\,n}$ is found as follows. Let d_j be the dimension of the subspace of $\mathbf{R}^n$ spanned by the vectors $A^i b_r$, $r = 1, ..., m$, $i \leqslant j$ where b_r is the r-th column of B. Let $e_j = d_j - d_{j-1}$, $d_{-1} = 0$. The partition corresponding to (A, B) is the dual partition of $(e_0, e_1, e_2, ..., e_n)$, i.e. $\kappa(A, B) = (e_0, e_1, ..., e_n)^*$. The numbers $\kappa_1 \geqslant ... \geqslant \kappa_m$ making up $\kappa(A, B)$ are called the Kronecker indices of (A, B). (Because the problem of classifying pairs (A, B) up to feedback equivalence, i.e. up to the action of F, is a subproblem of the problem of classifying pencils of matrices studied by Kronecker: to (A, B) one associates the pencil $(A - sI\ B)$. The partition $(e_0, ..., e_n)$ corresponds to the dimensions of the filtration of *controllability* subspaces.

Let θ_κ be the orbit of F acting on $L^{cr}_{m,n}$ labeled by κ. Then a second theorem, noted by a fair number of people independently of each other (Byrnes, Hazewinkel, Kalman, Martin, ...), but never yet published, states that $\bar{\theta}_\kappa \supset \theta_{\kappa'}$ iff $\kappa > \kappa'$. Some of the control theoretic implications of this are contained in Martin [32].

2.9. *Vectorbundles over the Riemann sphere.* Let E be a holomorphic vectorbundle over the Riemann sphere $S^2 = \mathbf{P}^1(\mathbf{C})$. Then according to Grothendieck [4] E splits as a direct sum of line bundles.

$$E \simeq L(\kappa_1) \oplus ... \oplus L(\kappa_m) \tag{2.10}$$

Where $L(i)$ is the unique (up to isomorphism) line bundle over $\mathbf{P}^1(\mathbf{C})$ of degree i, $L(i) = L(1)^{\otimes i}$, $i \in \mathbf{Z}$, where $L(1)$ is the canonical very ample bundle of $\mathbf{P}^1(\mathbf{C})$. Thus each holomorphic vectorbundle E over $\mathbf{P}^1(\mathbf{C})$ defines a m-tuple of integers $\kappa(E)$ (in decreasing order). The bundle is called positive if $\kappa_i(E) \geqslant 0$ for all $i = 1, ..., m$. Concerning these positive bundles there is now the following degeneration result of Shatz [19]. Let E_t be a holomorphic family of m-dimensional vectorbundles over $\mathbf{P}^1(\mathbf{C})$. Then for all small enough t, $\kappa(E_t) > \kappa(E_0)$. And inversely if $\kappa > \kappa'$ then there is a holomorphic family E_t such that $\kappa(E_\tau) = \kappa$ for t small $t \neq 0$ and $\kappa(E_0) = \kappa'$.

2.11. *Orbits of Nilpotent Matrices.* Let N_n be the space of all $n \times n$ complex nilpotent matrices. Consider $\mathbf{SL}_n(\mathbf{C})$ or $\mathbf{GL}_n(\mathbf{C})$ acting on N_n by similarity, i.e.

$$A^S = SAS^{-1}, \left(A \in N_n, S \in \mathbf{GL}_n(\mathbf{C})\right).$$

By the Jordan normal form theorem the orbits of this action are labelled by partitions of n. Let $O(\kappa)$ be the orbit consisting of all nilpotent matrices similar to the one consisting of the Jordan blocks $J(\kappa_i)$, $i = 1, ..., m$, where $J(\kappa_i)$ is the κ_i

$\times\ \kappa_i$ matrix with 1's just above the diagonal and zeros everywhere else. Then the Gerstenhaber-Hesselink theorem says that $\overline{0(\kappa)} \supset 0(\kappa')$ iff $\kappa < \kappa'$. (Note the reversion of the order with respect to the result on orbits described in 2.6. above.)

3. GRASSMANN MANIFOLDS AND CLASSIFYING VECTORBUNDLES

In order to describe how the various manifestations of the specialization order are connected to each other we need to define Grassmann manifolds, the classifying vectorbundles over them and their Schubert cell decompositions (in section 4 below).

3.1 *Grassmann Manifolds.* Fix two numbers $m, n \in \overline{\mathbf{N}}$. Then the Grassmann manifold $\mathbf{G}_n(\mathbf{C}^{n+m})$ consists of all n-dimensional subspaces of $\mathbf{C}^{n+m}$. Thus for example $\mathbf{G}_1(\mathbf{C}^{m+1})$ is the m-dimensional complex projective space $\mathbf{P}^m(\mathbf{C})$. Let $\mathbf{C}_{reg}^{n\times(n+m)}$ be the space of all complex $n \times (n+m)$ matrices of rank n. Let $\mathbf{GL}_n(\mathbf{C})$ act on this space by multiplication on the left. Then the quotient space $\mathbf{C}_{reg}^{n\times(n+m)}/\mathbf{GL}_n(\mathbf{C})$ is $\mathbf{G}_n(\mathbf{C}^{n+m})$. The identification is done by associating to $M \in \mathbf{C}_{reg}^{n\times(n+m)}$ the subspace of $\mathbf{C}^{n+m}$ generated by the rows of M.

$\mathbf{G}_n(\mathbf{C}^{n+m})$ inherits a natural holomorphic manifold structure from $\mathbf{C}^{n\times(n+m)}$. For a detailed description of $\mathbf{G}_n(\mathbf{C}^{n+m})$ see e.g. [16] or [23].

3.2. *The Classifying bundle.* We define a holomorphic vectorbundle ξ_m over $\mathbf{G}_n(\mathbf{C}^{n+m})$ as follows. For each x let the fibre over x, $\xi_m(x)$, be the quotient space $\mathbf{C}^{n+m}/x$. More precisely define the bundle η_n over $\mathbf{G}_n(\mathbf{C}^{n+m})$ by

$$\eta_n = \{(x, v) \in \mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m} \mid v \in x\} \tag{3.3}$$

with the obvious projection $(x, v) \mapsto x$. Then ξ_m is the quotient bundle of the trivial vectorbundle $\mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m}$ over $\mathbf{G}_n(\mathbf{C}^{n+m})$ by η_n. Both ξ_m and η_n can be used as universal or classifying bundles (cf. [16] for η_n as a universal bundle). Let E be an m-dimensional vectorbundle over a complex analytic manifold M. Let $\Gamma(E) = \Gamma(E, M)$ be the space of all holomorphic sections of E, i.e. the space of all holomorphic maps $s : M \to E$ such that $ps = id$, where $p : E \to M$ is the bundle projection. The universality, or classifying, property of ξ_m in the setting of complex analytic manifolds now takes the following form. Suppose $V \subset \Gamma(E)$ is an $(n+m)$-dimensional subspace such that for each $x \in M$ the vectors $s(x)$, $s \in V$ span $E(x)$, the fibre of E over x. Now identify $V \simeq \mathbf{C}^{n+m}$ and associate to $x \in M$